1. BASIC CONCEPTS AND TOOLS

In this course, we will study the dynamics of fluids, that is, hydrodynamics (HD). What, in this context, is a fluid? For the purposes of this course, there are three answers. (1) A normal, everyday liquid – such as water or molasses. The equation of state will depend on the substance; however, in most applications we will assume the density of the liquid is constant. (2) A diffuse gas – such as the atmosphere. The density of such a gas is definitely *not* constant; the relevant equation of state is the ideal gas law. (3) An ionized plasma – such as in laboratory plasma experiments, the upper layer's of earth's atmosphere, and nearly all astrophysical situations. The ideal gas law also applies here. In addition, the free charges in plasmas allow current flow; we must extend our subject matter to magnetohydrodynamics (MHD).

We will generally assume a fluid can be described in terms of macroscopic quantites. Our three basic variables will be the density ρ, the pressure p, and the velocity $\mathbf{v}$. We will often also use an equation of state, relating p and ρ. We will occasionally need transport: most often the viscosity ν, also thermal conductivity κ, and the electricial resistivity η. These are often derivable from the microphysics, but are usually considered as given constants for a particular problem.

A few comments are in order before we get into specifics.

• My *units* are cgs, not SI. This matters very little for normal fluid mechanics, but is critical for MHD. I will try also to introduce SI variants of basic electrodynamic terms and equations as they come up.

• My *notation* is driven somewhat by the vagaries of TEX. *vectors* are denoted by bold face roman, *e.g.*, $\mathbf{A}$; and for greek, either by bold (*e.g.*, $\boldsymbol{\omega}$, or by over-arrows (*e.g.*, $\vec{\omega}$). *Tensors* are more problematic: possibilities are direct subscripts, T_{ij}; $\vec{\vec{T}}$ is possible, $\mathbf{ab}$ for a product of two vectors, and occasionally bold face capitals (as in $\mathbf{P}$).

A. Kinematics: How to Describe a Flow

Visualizing a complex flow field can be important in understanding a problem. The most common concept is the *streamline*. This is a line (one of an infinite set of lines) that is everywhere tangent to the velocity vector in the flow. For instance, let $d\mathbf{s} = (dx, dy, dz)$ be the element of arc length along a steamline; and let $\mathbf{v} = (v_x, v_y, v_z)$ be the velocity vector at that point. The streamline is defined by

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \tag{1.1}$$

If the velocity field is specified, the streamline can be found directly from (1.1). In addition, (1.1 is equivalent to $d\mathbf{s} \times \mathbf{v} = 0$; no fluid crosses the streamline. We can also visualize a *streamtube*: a tubular region within the fluid, bounded by a set of streamlines (think of them as pink). Because streamlines can't intersect, the same streamlines follow the streamtube everywhere along its length.

Two other concepts are sometimes used. The *path line* is the trajectory of a fluid particle of fixed identity, over a period of time. (Paint a single fluid element red, say, and take a series of photos which display its location as time passes.) The *streakline*, through a fixed spatial point, is defined as the current location of all fluid particles which have passed through that point at some previous time. (Picture injecting dye at a fixed point in a flow, and taking one photo at some later time. The dye traces out the streakline). All three of these "lines" are the same in a steady flow; in a time-varying flow they will differ.

STREAM FUNCTION

Now consider a special case, when the flow is incompressible. That means constant density, and turns out to be equivalent to $\nabla \cdot \mathbf{v} = 0$ (we'll prove this immediately below). Now, we know from E&M that a vector field with zero divergeance can be written as the curl of some other vector: $\mathbf{v} = \nabla \times \mathbf{A}$, say. This isn't necessarily useful ... unless the flow field is two-dimensional. In that case, the vector potential $\mathbf{A}$ has only one non-zero component. To be specific, consider Cartesian coordinates, and assume the flow field only depends on (x, y). We can then write the (vector) potential as $\mathbf{A} = (0, 0, \psi)$, for some function $\psi(x, y)$; and the velocity field is

$$\mathbf{v} = -\nabla \times (\psi \hat{\mathbf{z}}) \tag{1.2}$$

This function $\psi(x, y)$ is called the *stream function*. We'll return to this in more detail in chapter 3, where we'll find that streamlines in the flow are also lines of constant ψ: each ψ value labels one particular streamline.

B. Mass Conservation: The Continuity Equation

This will be one of our most basic tools. Consider an arbtrary volume of fluid, V, bounded by a closed surface, A; let the surface have an outward normal, $\hat{\mathbf{n}}$. The mass within this volume is $\int_V \rho dV$, if ρ is the mass density. The net rate of change of this mass is

$$\frac{d}{dt}\int_V \rho dV \quad ;$$

if there are no sources or sinks of matter, this quantity must equal zero. Now, there are two ways this integral can change with time. (i) there can be intrinsic variation of ρ, $\partial\rho/\partial t \neq 0$; or (ii) there can be flow into or out of the volume, at a rate $\rho\mathbf{v}\cdot\hat{\mathbf{n}}$ per surface area. The sum of (i) and (ii) must balance out to zero:

$$\frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial\rho}{\partial t}dV + \int_A \rho\mathbf{v}\cdot\hat{\mathbf{n}}dA = 0 \quad (1.3)$$

But the surface integral can be written as $\int_A \rho\mathbf{v}\cdot\hat{\mathbf{n}}dA = \int_V \nabla\cdot(\rho\mathbf{v})dV$. Since V is arbitrary, we can set the integrand to zero, and we get the differential form of this basic equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad (1.4)$$

This is, of course, the continuity equation, applied to mass conservation.

LAGRANGIAN DERIVATIVE

Note, the two terms in the above expression describe the two "intrinsic" ways in which the mass in the elemental volume can change. We collect them as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \quad (1.5)$$

and with this, the continuity equation becomes

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \quad (1.6)$$

C. Momentum Conservation: Euler's Equation

Consider again our surface A, enclosing volume V. The momentum within this surface is $\int_V \rho\mathbf{v}dV$. The net rate of change of this quantity again must reflect intrinsic ($\partial/\partial t \neq 0$) variation and advection (flow across the surface). Thus, we write the net rate of change of momentum as

$$\int_V \frac{\partial}{\partial t}(\rho\mathbf{v})dV + \int_A (\rho\mathbf{v})\mathbf{v}\cdot\hat{\mathbf{n}}dA = \int_V \frac{\partial}{\partial t}(\rho\mathbf{v})dV + \int_V \nabla\cdot(\rho\mathbf{v}\mathbf{v})dV \quad (1.7)$$

In the second expression, we have used Gauss's law for tensors (noting that $\rho\mathbf{v}\mathbf{v}$ is a second-rank tensor).

Now, the net rate of change of momentum in the volume must be equal to the net force exerted on the volume. We consider external forces which act throughout the volume ("body" forces, such as gravity, electomagnetism, bouyancy, radiation pressure if the fluid is optically thin; we let $\mathbf{g}$ be the net force per mass), and also the force exerted on the surface by the fluid outside V. The net force on the volume V is, then,

$$\int_V \rho\mathbf{g}dV - \int_A p\hat{\mathbf{n}}dA = \int_V \rho\mathbf{g}dV - \int_V \nabla p dV \quad (1.8)$$

where we have again used vector identities in the last step. This balance, (1.7) and (), can then be written in differential form,

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = -\nabla p + \rho\mathbf{g} \quad (1.9)$$

This is our basic force equation, known as *Euler's equation*. Note that the $\mathbf{v}\mathbf{v}$ term is a tensor.[1]

It is conventional to simplify this, by expanding the derivatives on the left hand side and using (1.4); we get

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \rho\mathbf{g} \quad (1.10)$$

Another version of Euler's equation holds is useful if there is no body force. To get to this, write the pressure in terms of the pressure tensor (for an isotropic gas pressure):

$$\mathbf{P} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} = p\delta_{ij} \quad (1.11)$$

and note that the divergance of a tensor is found (in Cartesian) by

$$(\nabla\cdot\mathbf{P})_i = \frac{\partial P_{ij}}{\partial x_j} \quad (1.12)$$

with the Einstein summation convention assumed.

With this notation, the Euler equation in the case of no body force can be written,

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + \mathbf{P}) = 0 \quad (1.13)$$

This, and (1.4), are *conservative forms* of the basic equations.

[1] In general, a "vector product" tensor $\mathbf{ab}$ expands out as

$$\mathbf{ab} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

1. CONTROL VOLUMES

We argued above that equating (1.7) and (1.7) will give an integral form of the momentum balance equation. This is true if the volume V and surface A are fixed (not themselves moving). It can also be useful to consider a moving volume (such as in a rocket problem). To specify, let $V^*(t)$ be the instantaneous control volume, and $A^*(t)$ be its area. Let $\mathbf{b}$ be the local velocity of the *boundary*. Our integral equations become

$$\frac{d}{dt}\int_{V^*(t)} \rho dV + \int_{A^*(t)} \rho(\mathbf{v}-\mathbf{b})\cdot\hat{\mathbf{n}}dA = 0$$

and

$$\frac{d}{dt}\int_{V^*(t)} \rho\mathbf{v}dV + \int_{A^*(t)} \rho\mathbf{v}(\mathbf{v}-\mathbf{b})\cdot\hat{\mathbf{n}}dA = \int_{V^*(t)} \rho\mathbf{g}dV - \int_{A^*(t)} p\hat{\mathbf{n}}dA$$

Exercise for the student: can you derive or justify these relations?

2. BERNOULLI'S RELATION

It might be comforting to prove that we can extract Bernoulli's relationship from what we have so far.

Start with Euler's equation, in the form (1.10). But now, note two useful facts. The first is that if the fluid is *barotropic* – that is if $p = p(\rho)$ only (as in an adiabatic gas), we have

$$\frac{1}{\rho}\nabla p = \nabla \int \frac{dp}{\rho} \tag{1.14}$$

(this can be verified using the chain rule; take $p = F(\rho)$, F being some function, and go from there). Thus, this term is a perfect differential. The second useful fact is that

$$\mathbf{v}\cdot\nabla\mathbf{v} = -\mathbf{v}\times\boldsymbol{\omega} + \nabla\left(\frac{1}{2}v^2\right) \tag{1.15}$$

(this is easiest to verify by expanding out in Cartesian coordinates). Thus, this term is also a perfect differential. The first term on the right hand side is written in terms of $\boldsymbol{\omega} = \nabla\times\mathbf{v}$, the local *vorticity* (more in chapter 4 on this). Specify the force to gravity, which can be expressed in terms of a potential: $\mathbf{g} = \nabla\Phi_g$. If we then consider steady flow, we can rewrite (1.10) as

$$\nabla\left[\frac{1}{2}v^2 + \int\frac{dp}{\rho} + \Phi_g\right] = \mathbf{v}\times\boldsymbol{\omega} \tag{1.16}$$

But now: the right hand side of (1.15) is normal to both the local flow field (that is normal to streamlines) and to the local vorticity $\vec{\omega}$. Thus, we have one form of *Bernoulli's relation*: in inviscid, steady flow, the term in brackets has zero gradient in the direction of the local velocity field. Thus, we have one version of Bernoulli's law:

$$\frac{1}{2}v^2 + \int\frac{dp}{\rho} + \Phi_g = \text{constant along streamline} \tag{1.17}$$

Further, in an adiabatic gas, $p \propto \rho^\gamma$ if γ is the adiabatic index (the ratio of specific heats). The second term simplifies, so that Bernoulli's relation for an inviscid adiabatic gas is

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} + \Phi_g = \text{constant along streamline} \tag{1.18}$$

Alternatively, in an incompressible fluid, ρ is constant, and the second term in (1.15) becomes simply p/ρ. Thus, for an incompressible fluid, Bernoulli's relation is

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \Phi_g = \text{constant along streamline} \tag{1.19}$$

D. Work in a Rotating Frame

Euler's equation effectively expresses force balance. It must therefore be modified in a rotating system, where we expect Coriolis and centrifugal forces. Recall your basic mechanics. We want to transform vectors from our (intertial) frame to a frame rotating at $\mathbf{\Omega}$. Let $\mathbf{r}$ be the usual vector from the coordinate origin to the observation point (polar coordinates), and let $\mathbf{R}$ be a vector from the rotation axis, perpendicular, to the observation point (cylindrical coordinates). The time derivative of a general vector, $\mathbf{P}$, transforms as

$$\left(\frac{d\mathbf{P}}{dt}\right)_i = \left(\frac{d\mathbf{P}}{dt}\right)_r + \mathbf{\Omega}\times\mathbf{P}$$

Thus, velocities and accelerations transform as

$$\mathbf{v}_i = \mathbf{v}_r + \mathbf{\Omega}\times\mathbf{r}\ ;$$

$$\mathbf{a}_i = \mathbf{a}_r + 2\mathbf{\Omega}\times\mathbf{v}_r + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = \mathbf{a}_r + 2\mathbf{\Omega}\times\mathbf{v}_r - \Omega^2\mathbf{R}$$

From this, we can write the incompressible Euler equation in a rotating frame (now dropping the r subscripts):

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \Omega^2\mathbf{R} - 2\mathbf{\Omega}\times\mathbf{v} \tag{1.20}$$

The two new terms are the centrifugal and Coriolis forces.

E. Dimensional Analysis

Much of the art of fluid mechanics is knowing which terms in the equations can be thrown away. We are guided in this by several dimensionless numbers. In this section, we need "characteristic" values of the velocity V, the length scale L, the gas pressure p, the density ρ, and the magnetic field B. In addition, transport coeffients come in: ν is the viscosity, η the magnetic diffusivity, and κ is the thermal conductivity (these quantities will be introduced in more detail later, as they are needed).

As an example of why we consider these dimensionless numbers, return to the Euler equation, but with the viscous term added (just now, take my word for it, we'll derive it in Chapter 3 and give it another name):

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \rho\nu\nabla^2\mathbf{v} \tag{1.21}$$

The last term, a second spatial derivative in $\mathbf{v}$, is the viscous stress term. Consider the magnitude of each of the three terms:

$$O\left(\rho\frac{V}{t}\right) \sim O\left(\rho\frac{V^2}{L}\right); O\left(\rho\frac{c_s^2}{L}\right); O\left(\rho\nu\frac{V}{L^2}\right)$$

In writing these estimates, we have introduced the *sound speed*, $c_s^2 = \partial p/\partial\rho$ (it will appear formally in Chapter 7). In addition, we have assumed we can find a *characteristic velocity* V and a *characteristic length* L. Now, we can form the ratio of the first (the inertial term) to the second (the pressure gradient):

$$\frac{D\mathbf{v}/Dt}{\nabla p} \sim O\left(\frac{V^2}{c_s^2}\right) \tag{1.22}$$

and, the ratio of the first term to the third (the viscous force):

$$D\mathbf{v}/Dt \sim O\left(\frac{VL}{\nu}\right) \tag{1.23}$$

Thus, the relative importance of the three terms in the Euler equation are determined by the dimensionless quantities V^2/c_s^2 and vL/ν. These two ratios are fundamental in hydrodynamics, and have their own names. A similar analysis of other terms in these and/or other equations, as they arise, leads to several other important dimensionless numbers. I list several of these here.

The two most important for hydrodynamics are:

- the **Reynolds number** is

$$\mathrm{Re} = \frac{VL}{\nu} \tag{1.24}$$

Viscosity ν is addressed and defined in Chapter 3. This measures the ratio of the inertial term to the viscous term in the force equation. Viscosity is important at low Re; flows are laminar, *i.e.*, we can ignore viscosity, at higher Re. Interestingly, the transition to turbulence occurs at very high Re (even though viscosity is critically important in turbulence; this is discussed later).

- the **Mach number** is

$$\mathcal{M} = \frac{v}{c_s} \tag{1.25}$$

if c_s is the sound speed (shown in §3 to be given by $c_s^2 = \partial p/\partial\rho$). An important limit is the *incompressible*, or *Boussinesq* limit, namely that the fluid density is constant (from 1.4, this is equivalent to $\nabla\cdot\mathbf{v} = 0$); this is justified for flows in which $\mathcal{M} \ll 1$.

The most important scaling numbers for MHD are

- the **plasma beta** is

$$\beta = \frac{8\pi p}{B^2} \tag{1.26}$$

and gives the ratio of the gas pressure to the magnetic pressure ($p_B = B^2/8\pi$).[2] In low-beta plasmas, the magnetic field dominates the dynamics; in high-beta plasmas the gas pressure dominates and magnetic effects are small.

- the **Magnetic Reynolds number** is

$$\mathrm{Rm} = \frac{LV}{\eta} \tag{1.27}$$

if η is the magnetic diffusivity (addressed in Chapter 13). Flows with high Rm have strong coupling of the B field to the flow; flows at low Rm have only a weak coupling.

- the **Alfven Mach number** is

$$\mathcal{M}_A = \frac{v}{v_A} \tag{1.28}$$

if v_A is the Alfven speed, a characteristic signal speed in a magnetized fluid (shown in chapter 13 to be given by $v_A = B/\sqrt{4\pi\rho}$).

In addition, several other dimensionless numbers appear here and there. Some that we may meet are

- the **Prandtl number** is

$$\mathrm{Pr} = \frac{\nu}{\kappa} \tag{1.29}$$

and measures the radio of viscous diffusion to thermal diffusion.

[2] The SIs expression is $\beta = 2\mu_o p/B^2$.

• the **Rayleigh number** is

$$\mathrm{Ra} = \frac{\alpha g \Delta T L^3}{\kappa \nu} \tag{1.30}$$

and measures the ratio of the bouyancy force to the stabilizing effects of diffusion. α is the thermal expansion coefficient.

• the **Ekman number** is

$$\mathrm{E} = \frac{\nu}{\Omega L^2} \tag{1.31}$$

for a system rotating at angular speed Ω. This measures the ratio of viscous to Coriolis terms in the force equation.

• the **Rossby number** is

$$\mathrm{Ro} = \frac{V}{\Omega L} \tag{1.32}$$

in a rotating system. It measures the ratio of inertial to Coriolos terms in the force equation.

• the **Hartmann number** is

$$\mathrm{Ha} = \frac{BL}{\sqrt{4\pi\rho\nu\eta}} \tag{1.33}$$

and measures the ratio of magnetic forces to diffusive forces in a MHD fluid. Some authors omit the 4π.

• the **Magnetic Prandtl number** is

$$\mathrm{Pm} = \frac{\nu}{\eta} \tag{1.34}$$

and measures the ratio of viscous diffusion to magnetic diffusion.

References

Much of this discussion is "just from me" – *i.e.*, I've pulled the topics together "out of my head". Good places to go for more discussion include Kundu (who's usually quite mathematical); Tritton (who's much better with words and concepts); or Faber (our text).

For the plasma/Coulomb scattering discussion, Spitzer's *Physics of Fully Ionized Gases* is the classic reference.

F. Appendix: When can we use hydrodynamics?

Finally, we need to consider when we can use the hydrodynamic (or magnetohydrodynamic) model. Fluid mechanics may seem the obvious choice when we are concerned with laboratory or terrestrial liquids; they are "clearly" well described by macroscopic quantities such as density, pressure, velocity. For gases, and particularly plasmas, however, the relevance of fluid mechanics is slightly subtler. In particular, some astrophysical gases and plasmas are very tenuous: when can we justify a fluid approach, and when must we consider the dynamics of individual particles?

The general answer is, HD or MHD can be used when both the interparticle distance (equal to (density)$^{-1/3}$) and the mean free path of fluid particles (molecules, atoms, ions) is small compared to any characteristic scale of the system. That is, these distances must be small compared to the overall scale of the system, and must also be small compared to the scale of any gradients ($\rho/\nabla\rho$, for instance, is the density gradient scale).

1. HARD SPHERE COLLISIONS

Thus, we need to look at the mean free path in fluids, gases and plasmas. To start, we recall the basics of hard-sphere collisions. If a "gas" of billiard balls, say, has a number density n and each particle has a random velocity v and a radius a, we define the collision cross section,

$$\sigma = \pi a^2 \tag{1.35}$$

From this we find the mean free path (the average distance between collisions),

$$\lambda \simeq \frac{1}{n\sigma} \tag{1.36}$$

and the mean time between collisions,

$$t_{coll} \simeq \frac{1}{n\sigma v}\ . \tag{1.37}$$

This last can be inverted to describe the collision rate per particle, $t_{coll}^{-1} \simeq n\sigma v$.

For hard spheres this analysis is straightforward, of course; they will not interact unless there is a direct "hit", and the geometrical cross section is the relevant one to describe energy exchange. Neutral atoms and molecules behave similarly, in that they need a very close hit; their cross sections can be calcalated from basic physics. Typical atomic cross sections $\sim 10^{-14}\mathrm{cm}^2$.

2. PLASMAS: THE COULOMB CROSS SECTION

There is another type of encounter which is important in plasmas: a long-range encounter between two particles which feel a $1/r^2$ Coulomb force. The particles never directly collide with each other; but the long-range scattering allows exchange of energy and momentum, and thus makes the system act like a fluid.

• Start with a single encounter, in which particle A (an electron, say) scatters on particle B (a proton, say; with $m_p >> m_e$, we can assume the proton stays at rest. Let the incoming particle have velocity v and mass m_e, and let in come in at impact parameter b. We can solve this problem exactly, from classical mechanics, and find the deflection angle, θ, and the resultant velocity and momentum changes, $\Delta\mathbf{p} = m\Delta\mathbf{v}$. Here, we will approximate this analysis.

• The net impulse on the electron will be $\Delta\mathbf{p} = \int \mathbf{F}(t)dt$, integrated over the collision. Now, the force is strong only when the two particles are close. Since they are close for a period of time $\Delta t \simeq 2b/v$, we can approximate $F \simeq e^2/b^2$ and $\Delta p \simeq 2Fb/v$. (Since we know the net deflection is perpendicular to the initial direction of motion, we can also drop the vector notation). This gives us the net energy gain per collision,

$$\Delta E = \frac{(\Delta p)^2}{2m_e} \simeq \frac{2e^4}{m_e b^2 v^2}$$

We want to extend this analysis, to find the net rate of energy exchange with the plasma. But the collision rate of our electron, with particles at impact parameter b is, (collisions/second) $= 2\pi nbv\ db$, we find the net energy exchange rate by integrating over all allowed b:

$$\frac{dE}{dt} = \int_{b_{min}}^{b_{max}} \frac{2e^4}{m_e b^2 v^2} 2\pi bnvdb = \frac{4\pi e^4 n}{m_e v} \ln\left(\frac{b_{max}}{b_{min}}\right) \quad (1.38)$$

• Now, we want to express this in terms of a cross section:

$$\frac{dE}{dt} = \frac{E}{t_{coll}} = nv\sigma_c E \quad (1.39)$$

This defines the Coulomb cross section, σ_c:

$$\sigma_c = 8\pi\left(\frac{e^2}{m_e v^2}\right)^2 \ln\Lambda \quad (1.40)$$

if $\ln\Lambda = \ln(b_{max}/b_{min})$ is defined as the Coulomb logarithm.

• The Coulomb logarithm depends on the largest and smallest impact parameters that are important (clearly, we cannnot integrate from $b_{min} = 0$ to $b_{max} = \infty$, since the integral in (1.38) would diverge). b_{min} is usually taken to be the distance corresponding to maximum energy transfer,

$$b_{min} \simeq \frac{e^2}{2m_e v^2} \quad (1.41)$$

b_{max} is less straightforward. A common choice is the Debye shielding length (the scale over which an extra charge causes charge separation in a plasma):

$$b_{max} \simeq \lambda_D = \left(k_B T/4\pi n e^2\right)^{1/2} \quad (1.42)$$

(k_B is the Boltzmann constant, and T is the temperature). Thus, the best choice of $\ln\Lambda$ clearly depends on the exact situation one is considering. Luckily, for our purposes, this is only a logarithmic uncertainty, and will not be critical for most of our calculations. The choices above, with typical astrophysical parameters, give $\ln\Lambda \simeq 10 - 20$, in almost any diffuse-matter setting.

• Numerically, for a thermal plasma with $\frac{1}{2}m_e v^2 \simeq k_B T$, the Coulomb cross section becomes,

$$\sigma_c \simeq 7\times 10^{-13}\frac{\ln\Lambda}{T_4^2}\ \mathrm{cm}^2 \quad (1.43)$$

where $T_4 = T/10^4$K; so that

$$t_{coll} \simeq 4\times 10^4 \frac{T_4^{3/2}}{n\ln\Lambda}\ \mathrm{sec} \quad (1.44)$$

and

$$\lambda \simeq 1\times 10^{12}\frac{T_4^2}{n\ln\Lambda}\ \mathrm{cm}\ . \quad (1.45)$$

• A useful short way to remember the Coulomb cross section is as follows. Similarly to the b_{min} estimate above, we can define an effective "radius", a_{eff}, by equating potential and kinetic energies:

$$\frac{e^2}{a_{eff}} = \frac{1}{2}m_e v^2 \quad (1.46)$$

and then, estimating $\sigma_c = 2\pi a_{eff}^2 \ln\Lambda$. This recovers the form of equation (1.40), and resembles the hard-sphere cross section, (1.35), "with a factor of $\ln\Lambda$ tacked on". In extending this to other examples, as we will do just below, the exact numerical factor that scales $\pi a_{eff}^2 \ln\Lambda$ cannot be recovered by this method of guessing; one would have to do a more formal analysis to get the correct order-unity numerical factor for each cross section.

3. COLLISIONLESS PLASMAS

Finally, a brief comment on a third limit. We saw above that Coulomb collisions can transfer energy and momentum between charges in a plasma, thus "tying the plasma together" and justifying our treating it in the fluid limit. In many applications, however, the Coulomb mean free path derived using (1.40) is large compared to the size of the system. Does this then mean that we must use a single-particle (kinetic) approach? Not necessarily. Such systems can and do act like fluids; one example is the solar wind, which has a very long Coulomb mean free path, and yet is observed to carry shocks. (We will see later that shocks are mediated by particle-particle collisions; their thickness is several mean free paths). How can this be? There are two likely causes:

- **Tangled magnetic fields.** A charged particle in a magnetic field undergoes gyromotion. It is thus constrained to move (nearly) along the field line; gyroradii are typically quite small. If the magnetic field is tangled on a scale small compared to the system size, the system can be "tied togeter" by this effect, justifying a fluid approximation.

- **Plasma Microinstabilites.** A plasma is subject to a wealth of microinstabilities – in which free energy (such as that of streaming charges) is converted to wave energy in the plasma. The plasma waves which are excited involve fluctuating electric and magnetic fields, which in their turn scatter the plasma charges. This effect can also "tie the system together"; even a low-level wave energy density can lead to a short mean free path for particle-wave scattering.