

10. ONE-DIMENSIONAL UNSTEADY FLOW

In chapter 7 we introduced simple waves which carry information about some disturbance in the fluid (such as a piston moving or a sudden “hit”). In Chapter 9 we studied the local structure of shock fronts. In this chapter we return to the arguments of chapter 7, formalize them and allow for waves that are not weak. That is, we will formally introduce characteristics to solve unsteady flow problems. One important application of this is to shock formation and propagation.

A. Shock Formation: The Physical Picture

In Chapter 4, we treated sound waves as small-amplitude perturbations, and found that they propagate at a characteristic speed, c_s , which depends only on the temperature of the fluid. In reality, however, any such wave has higher density at its “center”, relative to its surroundings, and has a smooth variation of density with position. Figure 10.1 shows a discretized version of this.

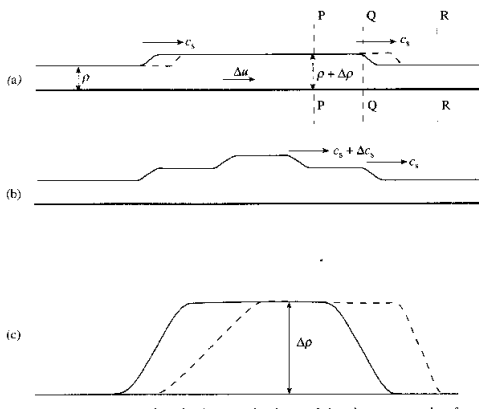


Figure 10.1. Density profiles for layers of higher density, propagating to the right. From Faber figure 3.5.

But the higher central densities lead to a higher central pressure and temperature (assuming an adiabatic wave which is the most likely). Thus the central signal speed will be higher than that at the edges; therefore the center will tend to overtake the leading edge. If the wave amplitude is large enough, an initially smooth waveform will be significantly distorted, and the wave will “break”. That is, the wave profile will become triple valued at its leading edge; this leads to the reforming of the edge in a shock discontinuity (as sketched in Figure 10.2).

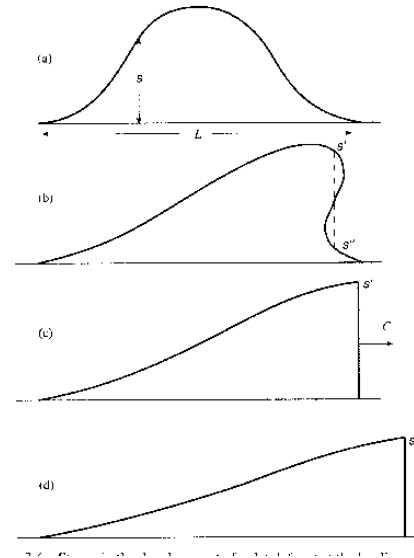


Figure 10.2. Schematic stages in the development of a wave which steepens into a shock. Note that all of the profiles contain the same area. From Faber figure 3.6.

SHOCK THICKNESS

Physically, shocks cannot be infinitely thin surfaces, although this is how they are treated mathematically. In practice, the fluid variables cannot change instantaneously; there must be a smooth transition between upstream and downstream. This defines a shock thickness, which is determined by viscosity, that is by interparticle collisions.

We can find a scaling argument for this: consider the balance between inertial and viscous forces, which must apply at the shock:

$$\rho v \frac{dv}{dx} \simeq \nu \frac{d^2 v}{dx^2} \quad (10.1)$$

This suggests the shock thickness, Δ , is determined by $\Delta \sim \nu/v$. But, the viscosity can be found from kinetic theory as $\nu \sim \lambda \langle v \rangle$, where λ is the mean free path and $\langle v \rangle$ is the mean particle speed. Thus, noting that $\langle v \rangle \sim c_s$ (check this out for an ideal gas if you don’t believe it!), we would estimate $\Delta \sim \lambda$. This makes some physical sense: the information that the flow conditions have changed cannot be communicated over a distance shorter than a collision mean free path.

This is not the entire picture, however. An ionized gas – a plasma – can support shocks which are much thinner than their collisional (Coulomb, *cf.* chapter 1) mean free path. The details of how these shocks work are still being argued about. There seems to be agreement, however, that these shocks are supported by microturbulence in the plasma: these small-scale fluctuations transfer the energy and momentum from upstream

to downstream, and convert ordered to random energy, just as interparticle collisions do in a shock in a neutral gas. The width of a collisionless shock is typically a fundamental plasma scale, such as the ion gyroradius or the length derived from the plasma frequency, $\sim c_s/\omega_p$.

B. The Method of Characteristics

In chapter 7 we introduced weak waves – waves of such small amplitude that the local conditions (density, pressure) are not significantly disturbed. These waves propagate at a constant speed, c_o , thus their loci in an (x, t) diagram are straight lines.

Our goal in this section is to extend this approach to the general case, rather than limiting ourselves to small-amplitude or subsonic flows. To do this, we first find the equations that describe the characteristics. Start with the 1D momentum and continuity equations (in a uniform channel, no A terms, and with no external body forces):

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (10.2)$$

and

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0 \quad (10.3)$$

But this latter can be written, using $\rho = \rho(p)$ and $dp = c_s^2 d\rho$, as

$$\frac{1}{\rho c_s} \frac{\partial p}{\partial t} + \frac{v}{\rho c_s} \frac{\partial p}{\partial x} + c_s \frac{\partial v}{\partial x} = 0 \quad (10.4)$$

Now, these last two can be added and subtracted to form two new equations,

$$\frac{\partial v}{\partial t} \pm \frac{1}{\rho c_s} \frac{\partial p}{\partial t} + (v \pm c_s) \left(\frac{\partial v}{\partial x} \pm \frac{1}{\rho c_s} \frac{\partial p}{\partial x} \right) = 0 \quad (10.5)$$

As Thompson notes, it appears doubtful that this is any simplification, but it is nonetheless. Define a new function $F = F(p, s)$ (where s is the specific entropy), which satisfies

$$F = \int_{p_o}^p \frac{dp}{\rho c_s} \quad (10.6)$$

where p_o is some (useful) reference state, and the integration is carried out at constant entropy. For an adiabatic perfect gas, the F integral can be done straightforwardly:

$$F = \int \frac{dp}{\rho c_s} = \int c_s \frac{d\rho}{\rho} = \int \frac{2dc_s}{\gamma - 1} = \frac{2}{\gamma - 1} c_s \quad (10.7)$$

Thus, this function F is simply a constant times the local sound speed. Now, the pressure derivatives can be written in terms of F , as

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial F}{\partial p} \frac{\partial p}{\partial t} = \frac{1}{\rho c_s} \frac{\partial p}{\partial t} \\ \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial p} \frac{\partial p}{\partial x} = \frac{1}{\rho c_s} \frac{\partial p}{\partial x} \end{aligned}$$

so that (10.5) can be written as

$$\left[\frac{\partial}{\partial t} \pm (v \pm c_s) \frac{\partial}{\partial x} \right] (v \pm F) = 0 \quad (10.8)$$

At this point, each equation (the “+” and “-” ones) contain only one derivative operator:

$$\begin{aligned} \frac{D^+}{Dt} &= \frac{\partial}{\partial t} + (v + c_s) \frac{\partial}{\partial x}; \\ \frac{D^-}{Dt} &= \frac{\partial}{\partial t} - (v - c_s) \frac{\partial}{\partial x} \end{aligned} \quad (10.9)$$

so that our two equations are, finally,

$$\frac{D^+}{Dt} (v + F) = 0; \quad \frac{D^-}{Dt} (v - F) = 0 \quad (10.10)$$

These equations turn out to be quite useful. We must interpret them physically, as follows. From (10.9) and (10.10), we see that the quantities

$$\begin{aligned} J^+ &= v + F = v + \frac{2}{\gamma - 1} c_s \\ J^- &= v - F = v - \frac{2}{\gamma - 1} c_s \end{aligned} \quad (10.11)$$

are constant along the lines described (and labelled) by

$$\begin{aligned} C^+ &: \frac{dx}{dt} = v + c_s; \\ C^- &: \frac{dx}{dt} = v - c_s \end{aligned} \quad (10.12)$$

These lines are called *characteristic lines*. They describe the paths of signals (sound waves) travelling forward or backwards in a flow, at the local sound speed. This, the quantity $v + F$ is constant for an observer travelling at velocity $v + c_s$, or on any plane perpendicular to the x axis which moves at $v + c_s$ (that is, with a forward, or positive, sound wave in a flow with velocity v . this is the *plus characteristic*, or the C^+ wave). Similarly, the quantity $v - F$ is constant for an observer travelling at velocity $v - c_s$ (that is, a backward or negative sound wave; the *minus characteristic* or the C^-

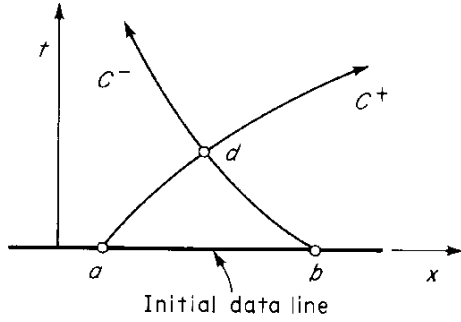


Figure 10.3. Characteristic lines, along which $J^+ = v + F$ and $J^- = v - F$ are constant. The solution at point d is uniquely determined by the initial data at points a and b . From Thompson figure 8.3

wave). These quantities, $J^\pm = v \pm F$, are called the *Riemann invariants*.

But these characteristics allow the simple solution of an initial value problem (conceptually; in practice numerical integration may be required). Refer back to Figure 10.4, which assumes we have a time-dependent problem in which initial values are specified over the range $a < x < b$. Consider point d in that figure. It is connected by a plus (forward direction) characteristic to point a , and a minus (backward direction) characteristic to point b . That means the quantity $J_a^+ = v + F$ at point d is the same at point a (since C^+ connects the two points), and the quantity $J_b^- = v - F$ is the same at point d and point b (since C^- connects them). Thus, we know the solution uniquely: $v = \frac{1}{2} (J_a^+ + J_b^-)$; $c_s = \frac{\gamma-1}{4} (J_a^+ - J_b^-)$.

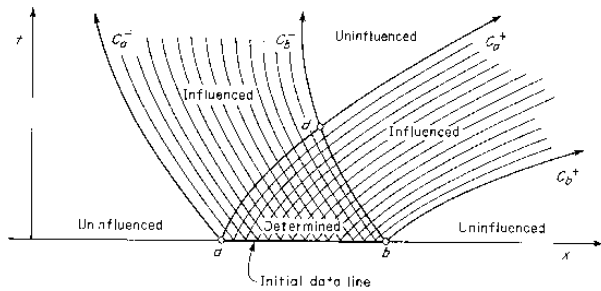


Figure 10.4. Regions of influence for the same initial-data problem as in Figure 10.4. From Thompson figure 8.4

Going further, referring to Figure 10.4, we can see the extent of the solution to this problem. The solution $v(x, t)$ is completely determined by the initial values in the region enclosed by the triangle abd . Outside this region, it is partly influenced in regions where one characteristic reaches; and it is totally underdetermined in

regions reached by no characteristic.

1. PISTON MOTION IN A CHANNEL

This may be made more concrete by the example of piston withdrawal, or advance, in a 1D channel. Let the piston position be described by $X(t)$; withdrawal has $dX/dt < 0$ and advance has the opposite sign.

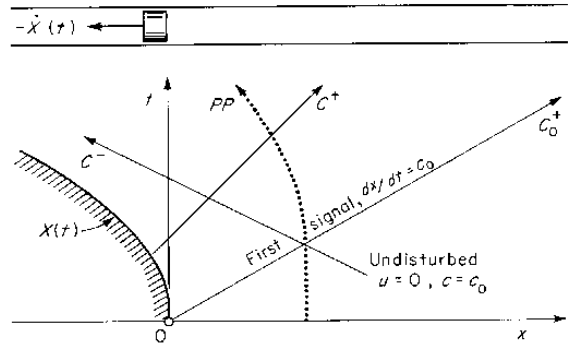


Figure 10.5. The geometry and characteristics associated with smooth piston withdrawal. PP labels the path of a particle starting at a particular point on the x axis. From Thompson figure 8.17.

We first notice that C^- characteristics come in from ahead, from the undisturbed flow region (which has $v = 0$, $c_s = c_{s,o}$). On these lines,

$$v(x, t) - \frac{2}{\gamma - 1} c_s = \frac{2}{\gamma - 1} c_{s,o} \quad (10.13)$$

This relation allows us to find v from $c_{s,o}$, or vice versa. Now, the C^+ characteristics connect the “influenced flow region” to the piston motion. On these lines,

$$v = \frac{dX}{dt}; \quad c_s = c_{s,o} + \frac{\gamma - 1}{2} \frac{dX}{dt} \quad (10.14)$$

In this latter, we have noted that a typical straight C^+ , which propagates into a uniform region, characteristic carries nearly constant c_s and v values. Thus, on C^+ we can identify v with dX/dt at the piston (assuming slow enough withdrawal that the fluid doesn’t separate); and we have used (10.13) as well. We note that the inverse slope of the C^+ characteristics is $v + c_s = c_{s,o} + dX/dt$. As this quantity becomes smaller, the characteristics become more and more vertical. We also note that the second expression in (10.14) shows that $c_s < c_{c,o}$ for a withdrawing piston; the expansion of the gas cools it. These relations (10.13) and (10.14) effectively solve the problem; they specify the values along each characteristic, and thus all regions of (x, t) space are addressed.

Piston advance can be treated in the same way, taking $dX/dt > 0$. The difference is that a shock forms ahead of the piston. The analysis above carries through just the same – we again note that $v + c_s = c_{s,o} + dX/dt$ is the inverse slope of the C^+ lines. But now, with $dX/dt > 0$, the slope decreases moving along the path of the accelerating piston: the characteristic lines must cross. As we will see next, a shock forms at the point where they first intersect.

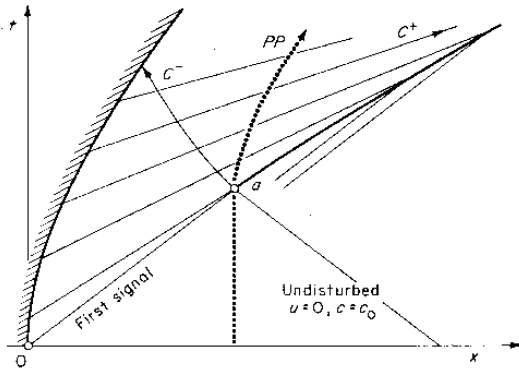


Figure 10.6. The geometry and characteristics associated with smooth piston advance. From Thompson figure 8.21.

2. CONNECTION TO SHOCK FORMATION

We argued in §10.1 that simple waves can steepen into shocks; this is based on the local propagation speed is an increasing function of the local density; so overdense regions move faster, and overtake the less dense regions ahead.

We can treat this more formally using characteristics. Let the initial wave be traveling in the $+x$ direction (a forward, C^+ , wave), into stationary gas with $p_0, \rho_0, c_{s,0}$ values far ahead of our disturbance.

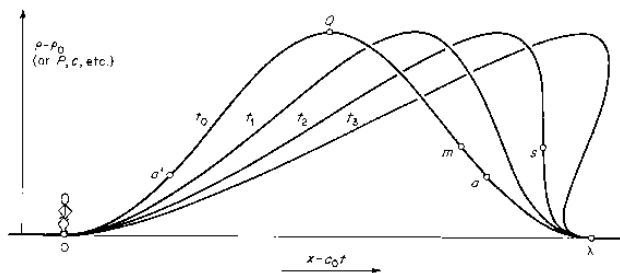


Figure 10.7. The geometry, and progressive distortion, of the steepening waveform described by characteristics in Figure 10.9. From Thompson Figure 8.10.

Now, the region around the disturbance includes C^- characteristics which come in from the uniform station-

ary region ahead of the wave. On these characteristics, $J^- = J_0^- = \text{constant everywhere}$, so that

$$J^- = v - \frac{2}{\gamma - 1} c_s = -\frac{2}{\gamma - 1} c_{s,o}; \quad (10.15)$$

and thus

$$v = \frac{2}{\gamma - 1} (c_s - c_{s,o}) \quad (10.16)$$

Thus, fluid within compression regions ($c_s > c_{s,o}$) moves in the $+x$ direction, the direction of travel of the wave. In addition, any portion of the C^+ wave must move with speed

$$v + c_s = \frac{\gamma + 1}{\gamma - 1} c_s - \frac{2}{\gamma - 1} c_{s,o} \quad (10.17)$$

Both of these connections verify the monotonic increase of wave speed with wave amplitude ($c_s - c_{s,o}$) which gives rise to the distortion of the wave into a shock. We can visualize this from characteristics also. The slope of each characteristic, in a (t, x) diagram, is the inverse of the local sound speed – so regions of higher overdensity have higher c_s , and thus lower slopes. Thus, characteristics from some region will *intersect* at some later time – shown as point s in the figure.

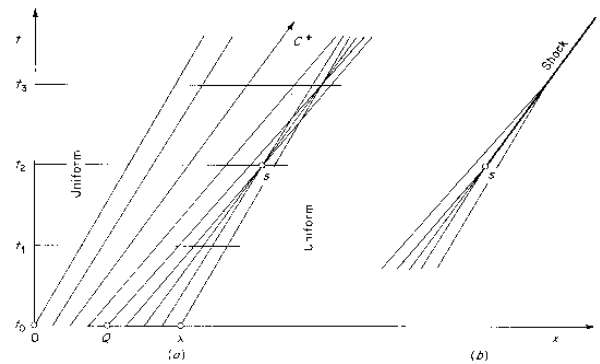


Figure 10.8. A characteristic diagram (wave diagram), showing characteristics intersecting at what becomes a shock. From Thompson figure 8.12

When is this time? In general, two nearby pieces of fluid have similar values of the quantity $u = v + c_s$; say they vary by δu (take this to be positive, just to pick a specific case). Referring to Figure 10.7, we see that the shock first forms, then, at the point where this derivative dx/du is a *minimum* – which occurs at the steepest point in the wave (that is along the leading edge, as in Figure 10.2).

Note, after the shock forms, our method of characteristics must be extended to include jumps at the shock itself.

3. TRAFFIC SHOCKS

Finally, let's consider a different approach: characteristics and shocks in traffic flow.

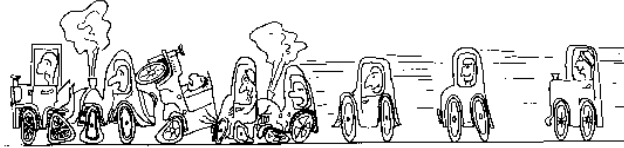


Figure 10.9. A traffic shock; from Thompson figure 7.3.

That is: consider the flow of cars along a one lane road (no passing!). We can take ρ as the density of cars (per length), v as their speed, and $q = \rho v$ as the traffic flow (what are the units here?). Conservation of cars is expressed by

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (10.18)$$

But this is, of course, just the continuity equation. Now, traffic-flow people simplify this by assuming the flow can be taken as a function only of the density: $q = q(\rho)$.¹ This allows the basic equation to be written

$$\frac{\partial \rho}{\partial t} + c(\rho) \frac{\partial \rho}{\partial x} = 0 \quad (10.19)$$

if $c(\rho) = \partial q / \partial \rho$.

Now, we know how to treat the equation (10.19): the solution ρ is constant along the characteristic lines

$$\frac{dx}{dt} = c(\rho) \quad (10.20)$$

Why? Consider the case when the “total derivative” is zero:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt} \frac{\partial \rho}{\partial x} = 0$$

But this is just equation (10.18) if the dx/dt term is given by (10.20). Thus, (10.19) shows that ρ is constant along the “path” in (x, t) space given by (10.20). That is, the lines on which ρ is constant, are *simple waves*, straight lines in the (x, t) plane.

Typically, the problem is set up in terms of initial conditions: $\rho(x, 0) = f(x)$, that is some function $f(x)$

describes the car density at $t = 0$. But each point in the (x, t) plane can be mapped back, along its local characteristic, to a starting point x_o . Thus, if things are single-valued, the solution can be found directly from $\rho(x, t) = f[x_o(x, t)]$. (If things are multivalued, that is characteristics from more than one x_o pass through a given (x, t) , then we find that a shock develops. Examples of both of these are given in the homework.

References

I'm mostly following Thompson here, also Faber on shock development. Traffic shocks can also be found as examples in PDE books.

¹ A typical choice assumes the velocity is $v(\rho) = V(1 - \rho/\rho_o)$, $\rho < \rho_o$. That is, a linear relation between density and speed, up to some maximum density ρ_o at which gridlock sets in and no further motion occurs.