11. TWO-DIMENSIONAL STEADY FLOW

We have studied transonic, one-dimensional flows, both time-steady and not. We now move to 2D, steady flows. There are two mathematical approaches that can be useful: linearized potential flow, and characteristics.

A. The Nature of Steady, two-dimensional flow.

As we have noted already, subsonic and supersonic flows behave very differently. A formal way to demonstrate this is through an extension of potential-flow techniques to a general, compressible flow. Consider a steady, irrotationa, inviscidl flow. Because $\nabla \times \mathbf{v} = 0$, we choose a velocity potential, $\mathbf{v} = \nabla\phi$. The continuity and Euler equations for steady flow are

$$\nabla \cdot (\rho\mathbf{v}) = 0\ ; \qquad (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = 0 \qquad (11.1)$$

Now, we note the useful fact, still for steady flow (whence?):

$$\frac{1}{\rho}\mathbf{v}\cdot\nabla p = -c_s^2\nabla\cdot\mathbf{v}$$

We can combine these results to get a second order equation in the velocity potential:

$$\frac{1}{2}\nabla\phi\cdot\left[\nabla(\nabla\phi)^2\right] = c_s^2\nabla^2\phi \qquad (11.2)$$

To be more explicit, we can pick Cartesian coordinates, and write

$$\frac{\partial^2\phi}{\partial x^2}(c_s^2-\phi_x^2)+\frac{\partial^2\phi}{\partial y^2}(c_s^2-\phi_y^2) = 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} \quad (11.3)$$

These equations (11.2) and (11.3) are general, but not very useful as they are horribly nonlinear in ϕ.

We can simplify it, however, by linearizing. In this section I describe one particular version of this: two-dimensional Cartesian flow. That is, consider a simple, unperturbed flow – for instance, a uniform flow U in the $\hat{\mathbf{x}}$ direction. Add a thin body, such as an airfoil, nearly aligned with the x-axis. This will perturb the flow slightly; we can specify the velocity potential to describe the *perturbed* flow. Going through the algebra (check Thompson, or Schreier, for the details), the linearized potential equation becomes

$$\left(U^2 - c_s^2\right)\frac{\partial^2\phi}{\partial x^2} = c_s^2\frac{\partial^2\phi}{\partial y^2} \qquad (11.4)$$

Or, in terms of the Mach number,

$$\left(1-\mathcal{M}^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (11.5)$$

But now, we note a significant difference for $\mathcal{M} > 1$ and $\mathcal{M} < 1$ flow. For subsonic flow, $\mathcal{M} < \infty$, (11.5) is essentially Laplace's equation, (3.3); so we can solve this linear problem with our potential flow techniques of chapter 2. We also note that such flows are totally governed by their boundary conditions.[1]

However, if $\mathcal{M} > 1$, the nature of equation (11.5) changes. In particular, it becomes a *wave equation*; and has as the general solution,

$$\phi(x,y) = F(x-\beta y) + G(x+\beta y) \qquad (11.6)$$

where $\beta^2 = \mathcal{M}^2 - 1$, and F, G are any function. This is getting back into the regime of characteristics – the flow is determined only by those regions nearby from which information can propagate.

We can restate this using some results from formal PDE theory. Equation (11.5) is a second order PDE; its nature depends depends on the sign of the $\left(1-\mathcal{M}^2\right)$ term. If $\mathcal{M} < 1$, the equation is *elliptic* – and does not have any real characteristics. If $\mathcal{M} > 1$, the equation is *hyperbolic*, and does have real characteristics. We can exploit those characteristics in the solution of steady, two-dimensional problems; which is the point of this chapter.

A final note: this breakdown into elliptic *vs.* hyperbolic does not depend on the linearization; Schreier demonstrates that the full potential system, (11.2) and (11.3) also changes type at $\mathcal{M} = \infty$.

B. Signal Propagation in Flows

If you don't care for such a mathematical approach, we can try a more physical one. Consider the *Mach construction* (due to Ernst Mach in an 1887 paper on supersonic projectiles). Let some object emit a steady signal while moving at some speed v through a fluid. (This signal might, for instance, be the simple fact that the object is moving and thus disturbing the local fluid). The signal propagates at c_s relative to the fluid. If the object is moving subsonically, the signals "converge" ahead of the object, and "diverge" behind it: this leads to the familiar Doppler effect. Note, all of space eventually receives the signals. The situation is different however, if the object is moving supersonically. In this case the spherical wavefronts emitted as the object moves, define an (upstream) cone of influence: fluid inside this cone can receive information about the motion, while

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[1] Remember all those problems you solved in electrostatics, for the electric potential, where the boundary conditions determined everything?

fluid outside cannot. The opening angle of this cone is

$$\sin \mu_M = \frac{c_s}{v} = \frac{1}{\mathcal{M}} \tag{11.7}$$

which is called the *Mach angle*. The surface bounding the range of influence of the object is called the *Mach surface* or *characteristic surface*; in two dimensions it becomes the *Mach lines*, or simply characteristics.

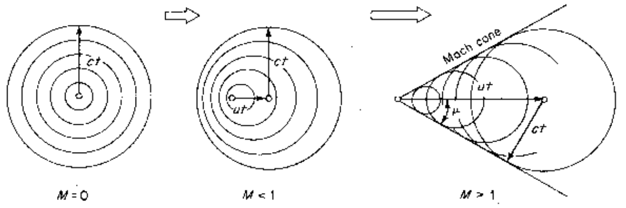

Figure 11.1. Repeating Figure 7.3, showing Mach's construction for the Mach cone and Mach angle. Note in this figure, the *flow* is moving to the right relative to the object; alternatively one can envision the object as moving to the left. From Thompson figure 5.2.

C. How Does Supersonic Flow Turn a Corner?

Now, we illustrate the use of characteristics in steady, supersonic flow with one example: flow around a corner.

Before we hit the math, however, let's think about the physics. Consider supersonic flow along a wall, and let the wall have a corner. Remember that information about the corner cannot propagate very effectively upstream, so the incoming fluid cannot "know" very easily about the corner. Just how, then, does the flow turn the corner?

To be specific, think about flow around a wall which turns a corner *away* from the flow, as in Figure 11.2b. Since we're working with supersonic flow, we can think back to the channel flow of Chapter 7, in particular flow into an expanding channel. The flow must expand, and being supersonic it must also speed up. This can be seen in terms of characteristics (also called *Mach lines* in this context). In the next section we will show that the Riemann invariants in this situation are given in terms of two angles: θ, the angle of the flow relative to some axis (say the wall before it turns), and $\delta(\mathcal{M}$, the Prandtl-Meyer angle defined in chapter 9. The invariants are $\theta \pm \delta(\mathcal{M})$. Our boundary ("initial") condition must be along the wall: the flow (streamline) must parallel the wall, at the wall. Thus, the characteristics must be straight lines which originate from the wall. Now, the plus (forward) characteristics have values of θ which go more and more negative, moving along the corner. Thus then tend to lean "forward". Minus characteristics come from ahead of a given point in the flow; at any point the intersection of plus and minus characteristics determines θ and $\mathcal{M}$ (the latter through the value of $\delta(\mathcal{M})$). In particular, the flow angle and speed are constant along any given characteristic. Thus, the flow must follow the bending of the Mach lines. This geometry forces the streamlines to diverge, so the flow must expand. Expanding a supersonic flow accelerates it – just as in chapter 7.

By the way: where does the energy for the acceleration come from?

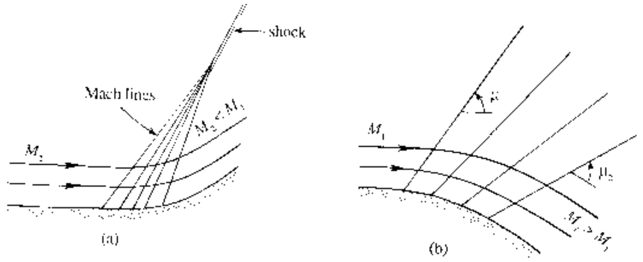

Figure 11.2. Flow turning corners, with streamlines and plus (forward) characteristics (Mach lines) shown. The left diagram shows flow into a bending wall. The streamlines come closer together as they follow the wall around, so the flow must compress and decelerate. A shock forms where the characteristics intersect. The right picture shows the opposite case, flow around a corner. The characteristics diverge, as does the flow; this expansion results in an acceleration. From Kundu figure 15.18

Now, consider the converse case: flow "into" a wall which bends, as in Figure 11.1a. The geometry forces the flow to be compressive, and thus (still being supersonic) it must slow down. Here, however, the corner-turning takes place through a set of shocks; it cannot be smooth. We can see this by drawing in characteristics again. The nature and angle of the characteristics follows the same rules as above; the angle of the wall sets the angles of the Mach lines which arise from it. But now, it's apparent that these lines must intersect: a shock will form. (Formally: think about characteristics intersecting. If "adjacent" lines carry different values of $\theta \pm \delta$, the values of θ and/or δ will not be uniquely defined at the intersection. That is not good. The fluid will respond by setting up a shock – an infinitely thin (well, nearly) jump between "front" and "back" properties.

By the way: when the flow decelerates, where does the energy go?

D. One example: Prandtl-Meyer flow

Now, let’s do this mathematically. A specific example of flow around corners is the case of flow around a sharp corner (as in Figure 3.14 of Faber). The flow expands through a “fan” of Mach lines centered at the corner, called the *Prandtl-Meyer expansion fan*. The mach number increases along a streamine through the fan, while the pressure falls along that streamine. Each mach line is inclined at μ_M to the *local* flow direction. Thus, supersonic flow turns via a sequence of standing oblique shocks.

This can be described in detail with characteristics. We could formally start with the second order PDE, (11.5), and find its characteristics. However, Thompson (his chapter 9) has an alternative approach which might be more illuminating. Following him, we search for characteristics by starting with the basic, 2D, steady-flow equations:

$$\begin{aligned} \nabla\cdot(\rho\mathbf{v}) = 0\,; \quad \nabla\times\mathbf{v} = 0 \\ \rho\nabla\left(\frac{v^2}{2}\right) + \nabla p = 0 \end{aligned} \tag{11.8}$$

We write these in streamline coordinates: s is the distance along the streamline, at local angle $\hat{\mathbf{v}}$; $\hat{\mathbf{n}}$ is the local normal, and θ is the flow angle (relative to some reference direction). Figure 11.2 illustrates the geometry. The zero-curl equation becomes

$$\frac{\partial v}{\partial n} + v\frac{\partial\theta}{\partial s} = 0 \tag{11.9}$$

which is the first equation of motion. For the moment, let’s work in cylindrical coordinates. The continuity equation becomes, in these coordinates,

$$\frac{1}{\rho}\frac{\partial\rho}{\partial s} + \frac{1}{v}\frac{\partial v}{\partial s} - \frac{\partial\theta}{\partial n} = -\frac{\sin\theta}{r} \tag{11.10}$$

This is the first equation of motion we want. Next, the momentum equation (in the s direction) becomes

$$v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{ds} = 0 \tag{11.11}$$

These last two combine, using $dp = c_s^2 d\rho$, as

$$\left(\mathcal{M}^2 - 1\right)\frac{1}{v}\frac{\partial v}{\partial s} + \frac{\partial\theta}{\partial n} = \frac{\sin\theta}{r} \tag{11.12}$$

which is the second equation of motion.

Now we use (11.9) and (11.11) to look for characteristics. In terms of the mach angle,

$$\tan\mu_M = 1/\sqrt{\mathcal{M}^2 - 1}$$

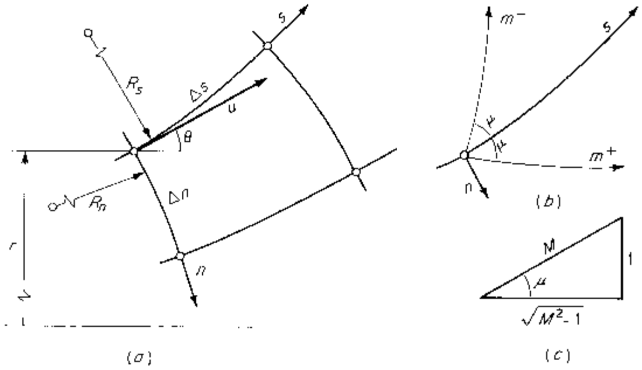

Figure 11.3. (a) Streamline coordinates for the characteristic solution. The flow velocity is $\mathbf{u}$, at angle θ to the x-axis; the distance along a streamline is s; $\mathbf{n}$ is the vector normal to the streamline. (b) Two Mach waves, m^+ and m^-, intersecting a streamline, (c) A geometrical way to remember the Mach angle. From Thompson figure 9.6

the two equations are

$$\begin{aligned} \tan\mu_M\frac{\sqrt{\mathcal{M}^2-1}}{v}\frac{\partial v}{\partial n} + \frac{\partial\theta}{\partial s} = 0 \\ \frac{\sqrt{\mathcal{M}^2-1}}{v}\frac{\partial v}{\partial s} + \tan\mu_M\frac{\partial\theta}{\partial n} = -\frac{\tan\mu_M\sin\theta}{r} \end{aligned} \tag{11.13}$$

But the first term in this second equation is $\partial\delta/\partial s$, if $\delta(\mathcal{M})$ is the Prandtl-Meyer function, defined in (9.27) above. Putting $\delta(\mathcal{M})$ into (11.13) where we can, and adding or subtracting the two, gives us our characteristic equations:

$$\begin{aligned} \left(\frac{\partial}{\partial s} + \tan\mu_M\frac{\partial}{\partial n}\right)(\theta + \delta(\mathcal{M})) = +\frac{\tan\mu_M\sin\theta}{r} \\ \left(\frac{\partial}{\partial s} - \tan\mu_M\frac{\partial}{\partial n}\right)(\theta - \delta(\mathcal{M})) = -\frac{\tan\mu_M\sin\theta}{r} \end{aligned} \tag{11.14}$$

These can be interpreted as follows. (Compare equations (10.10) in our earlier discussion of characteristics.) At any point along the streamline, two characteristics (a.k.a. Mach waves) can be found. They head out at angles $\pm\mu_M$ relative to $\mathbf{v}$, which defines the local streamline. (These are the analogs of $C^\pm$ in chapter 10).

But the left hand side of equations (11.14) are just the derivatives of $\theta \pm \delta(\mathcal{M})$ along the lines $m^\pm$. To see this, recall that (by the chain rule) the derivative of some function F along m^+ is

$$\frac{dF}{dm^+} = \frac{\partial F}{\partial s}\frac{ds}{dm^+} + \frac{\partial F}{\partial n}\frac{dn}{dm^+}$$

But, $ds/dm^+ = \cos\mu_M$ and $dn/\partial m^+ = \sin\mu_M$; so that

$$\frac{dF}{dm^+} = \cos\mu_M \left(\frac{\partial F}{\partial s} + \tan\mu_M \frac{\partial F}{\partial n} \right)$$

A similar result obtains for derivatives along m^-. Thus, the two characteristic equations for steady, two-dimensional flow can be written,

$$\begin{aligned} \frac{d}{dm^+}\left(\theta + \delta(\mathcal{M})\right) &= +\frac{\sin\mu_M \sin\theta}{r} \\ \frac{d}{dm^-}\left(\theta - \delta(\mathcal{M})\right) &= -\frac{\sin\mu_M \sin\theta}{r} \end{aligned} \tag{11.15}$$

Now, there are more general ways to treat characteristic problms, if the derivative of some function along a characteristic is not zero.[2] For now, let's switch back from cylindrical to plane geometry: that means take $r \rightarrow \infty$ in (11.15). This gives

$$\begin{aligned} \theta + \delta(\mathcal{M}) &= \text{constant} \qquad \text{on} \qquad m^+ \\ \theta - \delta(\mathcal{M}) &= \text{constant} \qquad \text{on} \qquad m^- \end{aligned} \tag{11.16}$$

and these equations are good analogs of (10.10). Thus, we have our characteristics ($m^{\pm}$) and our invariants [$\theta \pm \delta(\mathcal{M})$].

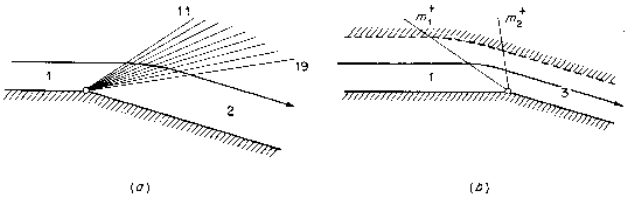

Figure 11.4. The geometry used in the example of a Prandtl-meyer solution. The wall makes a sudden turn by angle Δ away from the initial flow direction. m^- characteristics starting from the sudden turn, are illustrated. From Thompson figure 9.13

Finally, then, we can use this formalism to describe Prandtl-Meyer flow around a corner. To pick an example, consider upstream flow at $\mathcal{M}_1$, meeting a corner that turns by some angle Δ. The flow angle θ has a discontinuity at the corner; the interesting characteristics, m^-, diverge from here.

Now, the entire flow field is covered by characteristics m^+ which originiate in the uniform, upstream region 1 (where $\theta \rightarrow 0$): they have

$$\theta + \delta(\mathcal{M}) = \delta_1$$

everywhere. The other characteristics have

$$\theta - \delta(\mathcal{M}) - 2\theta - \delta_1 \qquad \text{on} \qquad m^-$$

where $\delta_1 = \delta(\mathcal{M}_1)$. From this, we get

$$\begin{aligned} \text{on} \quad m^- \ : \quad \theta = \text{constant} \qquad \text{and} \\ \delta(\mathcal{M}) = \delta(\mathcal{M})_1 = \theta = \text{constant} \end{aligned} \tag{11.17}$$

Thus, each m^- characteristic is a straight line, with constant $\delta(\mathcal{M})$ and θ values everywhere along it. There will then be a first intersting m^- characteristic, on which $\theta = 0^\circ$ and $\delta(\mathcal{M}) = \delta_1 = \delta(\mathcal{M}_1)$; and a last intersting one, on which $\theta_2 = -\Delta$, and $\delta_2 = \delta_1 + \Delta$; the final $\mathcal{M}$ value can be found from this, through the Prandtl-Meyer function.

References

Mostly from Schreier and Thompson, here.

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[2] We didn't bother with this in chapter 10, feel free to bug a mathematician if you're interested.