12. SIMILARITY SOLUTIONS

Similarity solutions can be a powerful method when they apply. Consider an example: a point explosion going off in a uniform atmosphere (such as a supernova explosion in a uniform interstellar medium). We expect a spherical shell will be driven out into the medium. How does this shell expand with time (what is the law for its size, $R(t)$?), and what is the gas structure within the shell?

We can skip ahead if we realize that there are only two scaling parameters in the problem: the total energy E, and the ambient density ρ_o. We expect the shell radius must involve only these two parameters, and the time t since the explosion. If we further guess that the expansion goes as a power law right (this turns out to be right; if we had made a bad guess our later analysis would show up the inconsistency), there is only one way to form a distance out of energy, density and time. That is, $R(t) \propto (E/\rho)^{1/5} t^{2/5}$. This must be the form of the dymamical law for the outer shell. We can skip further ahead: the *internal* structure of the expanding bubble must also be described in terms of this scaling law. For instance, r must appear only in combination as $r/R(t)$; velocity only as $v/\dot{R}(t)$; and so on.

Putting this scaling into the basic dynamical equations looks rather horrible, but actually leads to a very useful simplification of the analysis. This is best done by example, and I present three in what follows. You will see that the art of similarity solutions lies in understanding the problem, physically, beforehand, so that you can make a good guess as to the scaling laws and similarity variables. As often in solving physics problems, hindsight is a great help.

A. Blast Waves: the Sedov-Taylor Solution

We start with our initial example, a spherical blast wave resulting from a point explosion.

Dump a quantity E of energy, instantaneously, into a perfect-gas atmosphere with density ρ_o. The energy appears in the gas as kinetic and internal energy, with a strong psherical shock propagating away from the origin. Let V_s is the shock velocity, and ρ_s, p_s and v_s are the conditions just behind the shock. The jump conditions (9.12) for a strong shock become

$$\begin{aligned} p_s &= \frac{2}{\gamma+1}\rho_o V_s^2 \\ \rho_s &= \frac{\gamma+1}{\gamma-1}\rho_o \\ v_s &= \frac{2}{\gamma+1}V_s \end{aligned} \tag{12.1}$$

We see from this that the only constant dimensional quantities which enter the problem are E and ρ_o; these along with r and t consitutue the independent variables of the problem. (The dependent variables are v, p, ρ).

We want to put the basic equations into a dimensionless form. The standard approach in similarity solutions is to search for a dimensionless variable, η, which rules the problem. In general we want

$$\eta = (\text{constant}) r^{\lambda} t^{-\mu} \tag{12.2}$$

with λ, μ and the constant to be determined from the specific problem. Next, we set up scalings of the other physical quantities:

$$v = rt^{-1}\tilde{v}(\eta)\ ; \quad p = r^{-1}t^{-2}\tilde{p}(\eta)\ ; \quad \rho = r^{-3}\tilde{\rho}(\eta) \tag{12.3}$$

Derivatives of all the quantities must also be transformed. For instance,

$$\frac{\partial}{\partial t} \rightarrow -\frac{\mu\eta}{t}\frac{d}{d\eta}\ ; \qquad \frac{\partial}{\partial r} \rightarrow -\frac{\lambda\eta}{r}\frac{d}{d\eta} \tag{12.4}$$

(Think: why?). When this is done, the PDE's from the basic equations are transformed into ODE's. If there are no independent physical scales in the problem (for instance, does the ambient density have a length scale? If so then we can't solve the problem this way), then these ODE's can be solved once and for all time, giving us the solution to the problem.

Returning to the blast wave problem, the arguments in the introduction give us the dimensionless combination of the basic variables:

$$\eta = \left(\frac{\rho_o}{E}\right)^{1/5} \frac{r}{t^{2/5}} \tag{12.5}$$

and this is the similarity variable for this problem. The outer shock must then be labelled by some value of η, say η_o; so that the shock position as a function of time is

$$R_s(t) = \eta_o \left(\frac{E}{\rho_o}\right)^{1/5} t^{2/5} \tag{12.6}$$

That is, we expect the radius of the outer shell to be close to our dimensional analysis guess, (12.6), but we can't be sure of the exact coefficient yet. From (12.6), the shell/shock velocity it

$$V_s(t) = \frac{2}{5}\frac{R_s}{t} = \frac{2}{5}\eta_o \left(\frac{E}{\rho_o}\right)^{1/5} t^{-3/5} \tag{12.7}$$

We will find that $\eta_o \simeq 1$.

We can go further and determine the gas structure inside the outer shock. We work with dimensionless forms of the basic variables:

$$\begin{aligned} \hat{v} &= \frac{5(\gamma+4)}{4}\frac{u}{r/t} \\ \hat{p} &= \frac{25(\gamma+1)}{8}\frac{p}{\rho_o r^2/t^5} \\ \hat{\rho} &= \frac{\gamma-1}{\gamma+1}\frac{\rho}{\rho_o} \end{aligned} \qquad (12.8)$$

and each of these are only functions of η. (Check back with equations 12.3). The boundary conditions of the problem are

$$\hat{v}(\eta_o) = 1 \ ; \qquad \hat{p}(\eta_o) = 1 \ ; \qquad \hat{\rho}(\eta_o) = 1 \ . \quad (12.9)$$

at the shock front. The gas flow behind the shock is, as usual, described by the basic three equations:

$$\begin{aligned} \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0 \\ \frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial r} + v\frac{\partial \rho}{\partial r} + \frac{2\rho v}{r} &= 0 \\ \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\ln\frac{p}{\rho^\gamma} &= 0 \end{aligned} \qquad (12.10)$$

After a decent bit of algebra, these can be written in terms of the "hatted" variables:

$$\begin{aligned} \hat{\rho}\,(2\hat{v}-\gamma-1)\frac{d\hat{v}}{d\eta} + (\gamma-1)\frac{d\hat{p}}{d\eta} &= \frac{1}{2\eta}\left[\hat{\rho}\hat{v}(5\gamma+5-4\hat{v}) - 4(\gamma-1)\hat{p}\right] \\ \left(\hat{v} - \frac{\gamma+1}{2}\right)\frac{1}{\hat{\rho}}\frac{d\hat{\rho}}{d\eta} + \frac{d\hat{v}}{d\eta} &= -\frac{3\hat{v}}{\eta} \\ \frac{d}{d\eta}\ln\frac{\hat{p}}{\hat{\rho}^\gamma} &= \frac{1}{\eta}\frac{5(\gamma+1)-4\hat{v}}{2\hat{v}-(\gamma+1)} \end{aligned} \qquad (12.11)$$

But now, the bottom line: these equations are coupled ODE's, in the basic variable η. They can be integrated directly (that means numerically), for a given value of γ. This is the Sedov-Taylor solution, whose behavior is shown in Figure 12.1.

The value of η_o is also found numerically, from conservation of energy. That is, the total energy must satisfy

$$E = \int_0^R \left(\frac{1}{2}\rho v^2 + \frac{1}{\gamma-1}p\right) 4\pi r^2 dr \qquad (12.12)$$

and in dimensionless terms, this is

$$1 = \frac{32\pi\eta_o^5}{25(\gamma^2-1)}\int_0^1 \left(\hat{\rho}\hat{v}^2 + \hat{p}\right)\xi^4 d\xi \qquad (12.13)$$

where $\xi = \eta/\eta_o$. Numerical evaluation, after the $\hat{p}(\eta)$ and $\hat{v}(\eta)$ solutions are found, gives η_o – which turns out to be close to unity ($\gamma = 1.34 \Rightarrow \eta_o = 1.000$; $\gamma = 1.40 \Rightarrow \eta_o = 1.0033$; $\gamma = 1.67 \Rightarrow \eta_o = 1.153$).

Finally, we should note that this Sedov solution does not apply over the entire life of the explosion. It describes the early phases, in which the blast energy E is

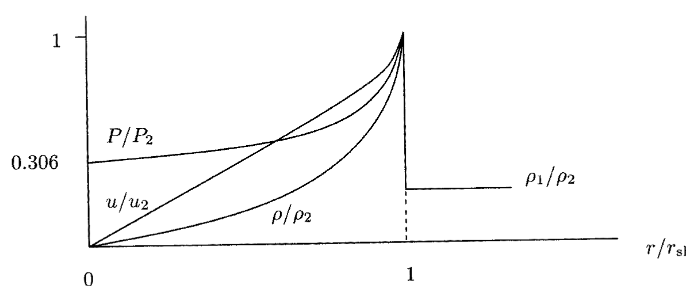

Figure 12.1. (Numerical) solutions for the gas structure inside the shock, as a function of the dimensionless variable $\eta = r/r_{sh}$, where $r_{sh} \simeq (Et^2/\rho_o)^{2/5}$. From Shu, Figure 17.3.

entirely contained in the explosion (Shu calls this the energy conservation phase). At later times, radiative losses become important,[1] the outer edge of the explo-

[1] Recall that radiative energy losses depend only on the microphysics, that is the density and temperature; they thus have a characteristic time scale which is not included in the analysis of this problem

sion gets denser and becomes a dense, thin shell. Past this point the expansion is governed by momentum conservation rather than energy conservation. . . which is another topic that we won't address here.

B. Prandtl-Meyer flow, revisited

In Chapter 11 we slogged through the characteristic-based solution to this problem; now, let's repeat the analysis using similarity methods. Here, we do not have an easy identification with a physical scale (there is no "radius of the shell" to consider). Instead, we guess that our fundamental variables must be angles (or, equivalently, the two cartesian coordinates (x, y) must only appear as the ratio x/y). Figure 12.2 has the geometry, in today's notation.

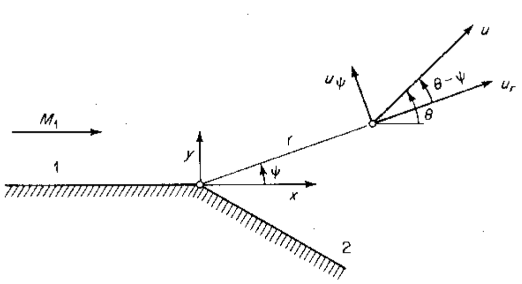

Figure 12.2. Geometry for the Prandtl-Meyer solution. From Thompson figure 10.1

The basic equations are, still, the three in (11.8). Rewriting them in polars, we have

$$\begin{aligned} \nabla \times \mathbf{v} &= \frac{\partial v_r}{\partial \psi} - v_\psi = 0 \\ \nabla \cdot (\rho \mathbf{v}) &= \rho v_r + \frac{\partial}{\partial \psi} \rho v_\psi = 0 \\ h + \frac{v^2}{2} &= \text{constant} \end{aligned} \tag{12.14}$$

(where we have used the definition of enthalpy h, from Chapter 6). We now note that there is no characteristic length scale in the problem, either in (12.14) nor in our boundary conditons (upstream quantities $\rightarrow p_1, v_1, \rho_1, \theta = 0$; downstream $\theta \rightarrow \theta_2$, the angle of the corner). Thus, the spatial variabiles x, y must appear only in the dimensionless form y/x. We pick the polar angle $\psi = \tan^{-1}(y/x)$ as the fundamental variable in the solution. This means there must be no dependence on the radial variable r.

Now: write $v_r = v\cos(\theta - \psi)$, and $v_\psi = v\sin(\theta - \psi)$. Recall, θ is the angle of the velocity vector, relative to the x axis, and ψ is the polar angle from the origin (the corner) to the point (r, ψ) under consideration. The irrotational equation becomes

$$\frac{d\theta}{d\psi} + \frac{1}{v}\frac{dv}{d\psi}\cot(\psi - \theta) = 0 \tag{12.15}$$

and, using the useful connection,

$$\frac{d\rho}{d\psi} = \frac{dh}{d\psi}\left(\frac{\partial \rho}{\partial h}\right)_s = v\frac{dv}{d\psi}\left(\frac{\partial \rho}{\partial h}\right)_s = -\frac{\rho v}{c^2}\frac{dv}{d\psi} \tag{12.16}$$

(where we used the chain rule; energy equation; and Gibbs relation, $dh = Tds + vdp$, to find $(\partial\rho/\partial h|_s = \rho/c^2)$. With this, the continuity equation becomes

$$\cot(\psi - \theta)\frac{d\theta}{d\psi} + \left(\mathcal{M}^2 - 1\right)\frac{1}{v}\frac{dv}{d\psi} = 0 \tag{12.17}$$

Equations (12.15) and (12.17) describe the system, as two ODE's in ψ.

Now, we can solve these two. Isolating $d\theta/d\psi$ and then eliminating it, we get

$$\cot(\psi - \theta) = \pm\sqrt{\mathcal{M}^2 - 1}) \tag{12.18}$$

But, since the RHS is the cotangent of the mach angle, we can write

$$\psi - \theta = \mu_M \tag{12.19}$$

This says that a radial line from the origin is a Mach line, *ie* the characteristic m^-. Using (12.18)), the equations of motion reduce to

$$\frac{d\theta}{d\psi} + \sqrt{\mathcal{M}^2 - 1}\frac{1}{v}\frac{dv}{d\psi} = 0 \tag{12.20}$$

This has the formal solution,

$$\theta + \int \sqrt{\mathcal{M}^2 - 1}\frac{dv}{v} = \text{constant} \tag{12.21}$$

But, the integral on the RHS is, again, the Prandtl-Meyer function (check equation 9.27). So, we have finally,

$$\theta + \delta(\mathcal{M}) = \delta(\mathcal{M}_1) \tag{12.22}$$

This solves the problem: the bending angle θ_2 is known, from the boundary conditions. Thus, the expansion reaches $\delta(\mathcal{M}_2) = \delta(\mathcal{M}_1) - \theta_2$, which allows one to find $\mathcal{M}_2$. Check back to chapter 11 – you can verify that this gets us to the same place as the characteristic analysis we used there.

C. Gravitational Collapse: the Shu Solution

A nice third example of this is the gravitational collapse of a self-gravitating cloud – such as a protostar. Frank Shu has developed this area extensively, and I take this (quite basic) analysis from his book.

What is the physical picture? Let's start with a static, isothermal sphere; take $a = (kT/m)^{1/2}$ as the sound speed. For this argument, let's consider the simple version with $\rho(r) \propto 1/r^2$, and ignore the unphysical divergeance at the origin. Imagine some perturbation, at $t = 0$, that makes the inside of the clour (protostar) slightly overdense. This will lead to an "inside-out" collapse of the protostar. As the expansion starts, for $r \sim 0$, we do not expect the outer regions to know about the expansion at all; an *expansion wave* will move outwards, at a speed a, as the collapse proceeds. This suggests a similarity variable, $x = r/at$. If we assume there is no important core (ignore the core discussed in (6.8) and following), then there is no physical scale length in the problem, and we can apply similarity methods.

Pick our basic equations, then. It is convenient to rewrite the continuity equation (1.4) in terms of mass shells, taking $M(r)$ as the mass within radius r:

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho \; ; \qquad \frac{\partial M}{\partial t} + v\frac{\partial M}{\partial r} = 0 \qquad (12.23)$$

For isothermal flow, the force equation is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{GM}{r} \qquad (12.24)$$

We convert to similarity variables:

$$\begin{aligned} \rho(r,t) &= \frac{\alpha(x)}{4\pi G t^2} \\ M(r,t) &= \frac{a^3 t}{G} m(x) \\ v(r,t) &= a v(x) \end{aligned} \qquad (12.25)$$

Putting these into (12.23) and (12.24) gives, after some algebra, an expression for the mass function,

$$m(x) = x^2 a[x - v(x)] \qquad (12.26)$$

and the two coupled ODEs,

$$\begin{aligned} \left[(x-v)^2 - 1\right]\frac{dv}{dx} &= \left[(x-v)a - \frac{2}{x}\right](x-v) \\ \left[(x-v)^2 - 1\right]\frac{1}{\alpha}\frac{d\alpha}{dx} &= \left[\alpha - \frac{2}{x}(x-v)\right](x-v) \end{aligned} \qquad (12.27)$$

These are the heart of this problem (compare the sets 12.11, or 12.15 and 12.17). Shu presents numerical solutions, which are illustrated in Figure 12.3.

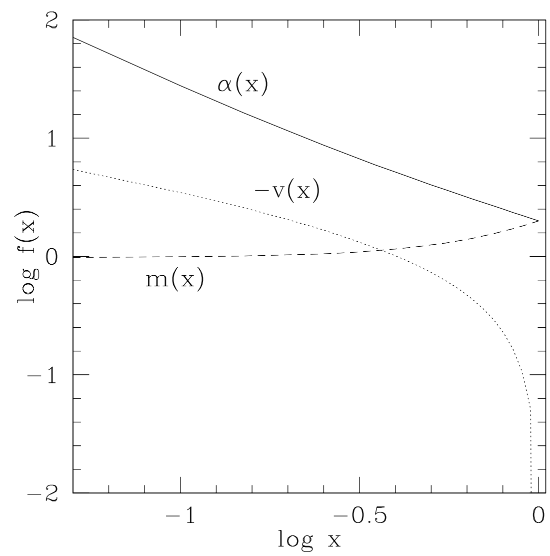

Figure 12.3. Self-similar solutions for Shu's collapse model. The solid line describes the density; the dotted line the infall velocity; and the dashed line, the mass function. From table 18.1 of Shu.

Applications and extensions of this may be discussed in class or in the homework (depending on Jean's mood and time).

References

The Sedov solution is the classical example here; I followed Thompson and Shore, but it's also in many other references. I took the Prandtl-Meyer solution from Thompson, and the Gravitational Collapse from Shu's book.