13. FOUNDATIONS OF MHD

In this chapter, we extend our fluid treatment to address current-carrying fluids in magnetic fields. The extra force term, from the $\mathbf{j} \times \mathbf{B}$ Lorentz force, adds – shall we say – an interesting richness to the problem. I am using several references for this part of the course, as I have not found one single book which covers the material usefully. Thus, some of my references come from labratory plasma work (confinement, stability, toroids), and others come from space physics and astrophysics (basic equations, flows, reconnection).

A. Remember your E&M?

We start with the full Maxwell:

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{E} &= 4\pi\rho_q
\end{aligned} \tag{13.1}$$

where ρ_q is the *charge* density. A couple of simplifications are common in MHD. First, note that we usually work with highly conductive fluids, that cannot support a finite space-charge density (why??). Thus, the $\mathbf{E}$ fields present are those induced, which are $O(vB/c) \ll B$ in subrelativistic flows (our limit in this course). Thus, $\partial\mathbf{E}/\partial t \sim O[(v/l)(vB/c)] = O[(cB/l)(v^2/c^2)]$, i.e. small, and we can also drop the displacement current in most of what follows. With these approximations, we can write the "reduced Maxwell equations:

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{E} &= 0
\end{aligned} \tag{13.2}$$

which will govern almost everything we do in this course.

1. CONDUCTIVITY AND OHM'S LAW: I

To close the (reduced) system we need to connect fields to currents. In most situations we can assume a vector Ohm's law, which in the absence of inductive effects would be written $\mathbf{j} = \sigma\mathbf{E}$, with scalar electrical conductivity σ. How do we include inductive effects here? In our reduced MHD, we can write $4\pi\mathbf{j} = c\nabla \times \mathbf{B}$. Now go to a frame moving at $\mathbf{v}$ (relative to the lab); and keep $v \ll c$ to avoid worrying about relativity. In that frame there will be a simple Ohm's law, $\mathbf{j}' = \sigma\mathbf{E}'$. But we know the fields transform as

$$\mathbf{E}' = \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \; ; \quad \mathbf{B}' = \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) \tag{13.3}$$

(recall $\gamma \simeq 1$ here). Now, we know that any charge density ρ_q in the lab frame will contribute an extra current $\rho_q\mathbf{v}$ in the comoving frame ... however with $\mathbf{E}$ small, and $\mathbf{v}$ also small, any extra $\mathbf{j}'$ is (v^2/c^2). We therefore have $\mathbf{j} = \mathbf{j}'$, and thus in the lab we can write

$$\mathbf{j} = \sigma\mathbf{E}' = \sigma\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{13.4}$$

We can also interpret (13.4) by noting that $\mathbf{E}'$ is the field felt by an observer (or an electron) moving with the fluid element.

2. CONDUCTIVITY AND OHM'S LAW: II

In a high density fluid collisions usually dominate particle motions, so that a scalar conductivity σ and the vector form of Ohm's law, (13.4), is enough. Occasionally, however, we need to extend the description to consider microphysical effects in a low density plasma. In particular, applying a $\mathbf{E}$ field to a magnetized plasma will result in currents *parallel* and *perpendicular* to the background field B_o. The details are in the Appendix; here, just note that the *Generalized Ohm's law* can be written as

$$\mathbf{j} = \sigma_o\mathbf{E}_{\parallel} + \sigma_{\perp}\mathbf{E}_{\perp} + \sigma_H\hat{\mathbf{n}} \times \mathbf{E} \tag{13.5}$$

Here, $\mathbf{E}_{\parallel}$ is the component of $\mathbf{E}$ along $\mathbf{B}$; $\mathbf{j}$, $\mathbf{E}_{\perp}$ is the component of $\mathbf{E}$ across $\mathbf{B}$; and $\hat{\mathbf{b}}$ is the unit vector along $\mathbf{B}$. The first term uses the normal (collisional) conductivity; the second uses the *Pederson* conductivity; and the last (usually the smallest) uses the *Hall conductivity*.

B. Field Evolution: Induction Equation

How does the field evolve? We use (13.4), but consider only the inductive part;[1] and we work with "reduced" Maxwell (13.2):

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{c}{4\pi\sigma}\nabla \times \mathbf{B} \tag{13.6}$$

[1] Why? because we don't expect any unshielded free charge to stick around ... right?

Alternatively, using vector algebra and defining the quantity $\eta = c^2/4\pi\sigma$, this becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \qquad (13.7)$$

which is often called *the induction equation*. Two comments on this equation.

• Compare this to equation (4.8), which describes the evolution of vorticity, in the barotropic limit. They are just the same: and so we should expect the magnetic field and the vorticity to behave similarly in hydrodynamic situations. One example is flux freezing, which we derive immediately below: it's the MHD analog of Kelvin's theorem. Another example is the existence of magnetic *flux ropes*: a field subject to fluid forces tends to bunch into linear, high-field regions (think of a tornado or a vortex line).

• Equation (13.7) has two important limits, as follows. When we worked with fluids, we noted that the behavior of a system is sensitive to the ratio of the advective to dissipative terms. We expressed this ratio in terms of the Reynolds number. The induction equation admits a similar separation of effects. We take the *magnetic Reynolds number*

$$\mathrm{Rm} = \frac{LV}{\eta} \qquad (13.8)$$

where L, V are characteristic length and velocity scales of the problem. This quantity again measures the radio of inertial/advection, to dissipative, terms.

1. IDEAL LIMIT: FLUX FREEZING

An imporant limit in MHD is the limit when $\mathrm{Rm} \gg 1$, called *ideal MHD* . This is the case of a highly conducting plasma. Here, the induction equationk (13.7), reduces to

$$\frac{\partial \mathbf{B}}{\partial t} \simeq \nabla \times (\mathbf{v} \times \mathbf{B}) \qquad (13.9)$$

while Ohm's law becomes

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \simeq 0 \qquad (13.10)$$

Consider, now, a closed curve (C) bounding a surface (S) which is moving with the plasma. The magnetic flux through S is $\Phi_B = \int \mathbf{B} \cdot d\mathbf{S}$; and its rate of change can be written

$$\frac{d\Phi_B}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) \qquad (13.11)$$

The first term describes intrinsic changes in $\mathbf{B}$, while the second describes advection of the field through the boundary. Now, Stokes' theorem rewrites this as

$$\frac{d\Phi_B}{dt} = \int_S \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{S} \qquad (13.12)$$

Now, (13.7) allows us to write this, generally, as

$$\frac{d\Phi_B}{dt} = \int_S \eta \nabla^2 \mathbf{B} \cdot d\mathbf{S} \qquad (13.13)$$

Thus, as $\eta \to 0$ (the perfect conductivity limit, an "ideal" plasma), Φ_B becomes a constant of the motion. That is, the magnetic flux through a loop which is "attached to the plasma", stays constant as that loop is stretched (or compressed) by the flow field. The mean magnetic field in that loop therefore decreases (or grows), in proportion to the change of loop area. This is *Flux Freezing*.[2]

Magnetic Field Lines? *(A discussion from Schmidt...)* The concept of a magnetic line of force is an abstraction. IN general no identity can be attached to these lines (they cannot be labelled in a varying field), nor can we speak of "motion" of field lines. In a perfect conductor, however, the concept of field lines becomes meaningful, due to flux freezing.

Consider a material line in the fliud (say a chain of labelled droplets, or particles painted pink), defined by intersecting material surfaaces. Choose these surfaces everywhere tangential to $\mathbf{B}$ at $t = 0$. The flux through both surfaces is therefore zero to start, and their intersection defines a field line at that point. Flux freezing guarantees that these surfaces continue to $\Phi_B = 0$ at any later time. Thus, their intersection continues to define a field line, in fact the same field line – it has become identifiable; labelling the material (painting it pink) has labelled the field line, and the local fluic velocity $\mathbf{v}(\mathbf{x}, t)$ is also the velocity of that section of the field line. *The field line is attached to – "frozen into" – the fluid.*

2. RESISTIVE LIMIT: FLUX ANNIHILATION

In a fluid with finite conductivity, flux freezing no longer holds. We can explore this by going to the other limiting case, when $\mathrm{Rm} \ll 1$. This is *diffusive limit.* If we simply ignore the advection term, equation (13.7) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} \qquad (13.14)$$

[2] Compare Kelvin's Theorem, from chapter 4; you will see that the math is identical, as is the nature and interperation of the result.

This describes the effect of Ohmic dissipation on the magnetic field; note that it is a standard diffusion equation.

Diffusion of field lines? We know how solutions to (18.4) behave: an initial field will decay on a timescale $\sim L^2/\eta$. Some authors discuss this in terms of field line "diffusion" or "slippage" out of the fluid. Remember that the density of field lines is related to the strength of the field; so a lower density of field lines, with time, should correspond to field lines "diffusing" out of the field. In particular, when η is finite, field lines are no longer tied to parcels of the plasma; some authors talk of field lines "moving through" the plasma in dissipative regions.

C. Fluid Equations: Lorentz force

The effect of the $\mathbf{B}$ field on the force equation is straightforward. We simply add the Lorentz force to the momentum equation:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{\mathbf{j}}{c} \times \mathbf{B} \tag{13.15}$$

Note, I have ignored viscosity here, as well as any external forces (such as gravity). Now: expand out the Lorentz force as

$$\begin{aligned} \frac{\mathbf{j}}{c} \times \mathbf{B} &= \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= -\frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} \end{aligned} \tag{13.16}$$

This is an important breakdown of the Lorentz force; it demonstrates that the field exerts a *magnetic tension* and a *magnetic pressure* on the fluid. The first term in (13.16) represents the gradient of a scalar pressure, $p_B = B^2/8\pi$. It appears in the momentum equation parallel to the fluid pressure....you can think of trying to compress a magnetic field, parallel to itself, with the field resisting the compression ("fighting back"). The second term in (13.16) is non-zero only if the field varies parallel to itself. A simple illustration is a curved field line. The curvature means there is a current flowing along the field line; the $\mathbf{j} \times \mathbf{B}$ force points inwards (relative to the curvature). Thus, curved field lines "want to straighten out"...Some authors combine both effects by describing magnetic field lines as "elastic bands within the fluid", which resist being stretched: either pushed together, or pulled transverse to their length.

D. Fluid Equations: Energetics

We need two items here: first recall the energetics of the fields, by themselves; then their effects on the energetics of the fluid.

1. ENERGETICS OF THE E AND B FIELDS

To remind yourself of this...go to Jackson.[3] Keep all of the terms – the full Maxwell's equations. Consider the quantity $\mathbf{E}\cdot\mathbf{j}$: it expands as

$$4\pi\mathbf{j}\cdot\mathbf{E} = c\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

But now, use the vector identity $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$, and Maxwell again, to get

$$-\mathbf{j}\cdot\mathbf{E} = \frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}$$

Now: we identify the the energy density in the fields, as $u_E = E^2/8\pi$; and $u_B = B^2/8\pi$. And we integrate the equation above over some volume V, with a surface S, giving

$$\begin{aligned} -\int_V \mathbf{j}\cdot\mathbf{E}dV &= \int_V \frac{d}{dt}(u_E + u_B)\,dV \\ &\quad + \int_S \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S} \end{aligned} \tag{13.17}$$

Thus: the first term is the rate of change of field energy in V. The second term – involving the Poynting flux, $\mathbf{S} = (c/4\pi)(\mathbf{E}\times\mathbf{B})$ – is the flow of EM energy through the surface. The last term, then, is the work done by the fields on the sources in the volume (and it can have either sign ... thinking about driving vs. dissipation .. I think). This can also be written in differential form:

$$\frac{d}{dt}(u_E + u_B) + \nabla\cdot\mathbf{S} = -j\cdot\mathbf{E} \tag{13.18}$$

which looks much like any other conservation equation.

2. ENERGETICS OF THE FLUID

Schmidt has a nice approach: I've redone it in cgs, keeping all the terms. To start, dot $\mathbf{v}$ into the force equation (13.15):

$$\begin{aligned} \rho\mathbf{v}\cdot\frac{D\mathbf{v}}{Dt} + \mathbf{v}\cdot\nabla p &= \frac{1}{c}\mathbf{v}\cdot\mathbf{j}\times\mathbf{B} \\ &= \frac{1}{4\pi}\mathbf{v}\cdot\left[(\nabla\times\mathbf{B}) - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right]\times\mathbf{B} \end{aligned} \tag{13.19}$$

where I've used the full Maxwell on the right. With some algebra, the terms on the LHS can be written:

$$\begin{aligned} &\rho\mathbf{v}\cdot\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla\cdot\left(\frac{1}{2}\rho v^2\mathbf{v}\right) \end{aligned} \tag{13.20}$$

[3] Preferably the 1st edition, where he works in in cgs.

and

$$\mathbf{v}\cdot\nabla p = \frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{\gamma}{\gamma-1}\nabla\cdot(p\mathbf{v}) \qquad (13.21)$$

where we've used $p \propto \rho^\gamma$ in this last; otherwise these are both general, for compressible fluids. So far, this is just the same as the first part of the course. Now, consider the last term, using vector algebra to reorganize each term:

$$\frac{1}{4\pi}\mathbf{v}\cdot\left[(\nabla\times\mathbf{B}) - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right]\times\mathbf{B} = -\frac{1}{4\pi}(\mathbf{v}\times\mathbf{B})\cdot(\nabla\times\mathbf{B}) + \frac{1}{4\pi c}(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial\mathbf{E}}{\partial t} \qquad (13.22)$$

Now ... use $\mathbf{v}\times\mathbf{B}/c = \mathbf{j}/\sigma - \mathbf{E}$; and $\mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{E} - \nabla\cdot(\mathbf{E}\times\mathbf{B})$; expand out and collect terms, to get

$$\frac{\mathbf{v}}{c}\cdot(\mathbf{j}\times\mathbf{B}) = \frac{1}{\sigma}j^2 - \frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}\right) - \frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B}) \qquad (13.23)$$

Collecting everything, we get

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \frac{1}{\gamma-1}p + \frac{1}{8\pi}(E^2+B^2)\right] + \nabla\cdot\left[\frac{1}{2}\rho v^2\mathbf{v} + \frac{\gamma}{\gamma-1}p\mathbf{v} + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right] = -\frac{1}{\sigma}j^2 \qquad (13.24)$$

This is still fully general; it combines the EM and fluid terms in a conservative form... We can now follow the usual MHD approach and drop E^2 in the $\partial/\partial t$ term (as Schmidt does). Also note that Parker writes the Poynting flux, in the MHD case, as $\mathbf{S} = \mathbf{B}\times(\mathbf{v}\times\mathbf{B})/4\pi = \mathbf{v}_\perp B^2/4\pi$, explicitly assuming ideal and an induction-only $\mathbf{E}$ field.

References

Once again, much of this basic material is "just from me". Possibly useful references might be Priest (who has a very good MHD introduction), also Woods and Davidson. I followed Schmidt for the energetics analysis, and cannot for the life of me remember where I first found the Generalized Ohm's law arguments (probably Chen's plasma book? ... sorry).

E. Appendix I: Conductivity notes

1. COLLISIONAL CONDUCTIVITY

First, simple collisional (parallel) conductivity, σ. We can find σ simply, in terms of the mean time between collisions, t_{coll}, or its inverse the collision rate, $\nu_{coll} = 1/t_{coll}$, as follows. Consider a free electron, in a plasma, subjected to an external electric field E. The net force on the particle can be estimated,

$$\mathbf{F}_{net} \simeq e\mathbf{E} - \frac{\Delta\mathbf{p}}{\Delta t} \qquad (13.25)$$

where $\Delta p/\Delta t$ is the mean rate of momentum change per collision. But if the charges have a net drift velocity v_D, we can estimate $\Delta p/\Delta t \sim m_e v_D/t_{coll}$; then, in a steady state we have $F_{net} \simeq 0$, so that

$$e\mathbf{E} \simeq n\mathbf{v}\nu_{coll} \qquad (13.26)$$

and the drift velocity must be $v_D = eEt_{coll}/m_e$. Next, we can use this in the (static) Ohm's law, to relate the conductivity to the drift velocity:

$$j = n_e e v_D = \sigma E \qquad (13.27)$$

where the second equality defines σ. Collecting everything, we end up with

$$\sigma = \frac{n_e e^2}{m_e\nu_{coll}} = \frac{\omega_p^2}{4\pi}\tau_{coll} \qquad (13.28)$$

In the last expression the *plasma frequency* is used: $\omega_p^2 = 4\pi ne^2/m$. The simplest physics is when the collision time is determined by Coulomb collisions in the plasma, as in chapter 1 (check §1.F). Alternatively, the collisions may be with microturbulence in the plasma ... in which case the collision rate usually has to be put in by hand (rather than calculated from first principles).

2. CROSS-FIELD CONDUCTIVITY

Consider a general situation in which $\mathbf{E}$ and $\mathbf{B}$ exist; there will be $\mathbf{E}_{\|}$ and $\mathbf{E}_{\perp}$ components, relative to $\mathbf{B}$. We can anticipate the three terms in the Generalized Ohm's law, as follows: (1) collisional conductivity, giving a field-aligned current proportional to $\mathbf{E}_{\|}$, as usual; (2) quasi-collisional conductivity, giving a cross-field current proportionao to $\mathbf{E}_{\perp}$; and (3) a transverse current, perpendicular to both $\mathbf{E}$ and $\mathbf{B}$, based on single-particle $\mathbf{E}\times\mathbf{B}$ drift.[4] Note that this drift is independent of particle charge and mass .. but we get a net current because the collision frequencies are not. There are two approaches, macro and micro.

• **macroscopic**. Now: just consider one species, still, and extend (**??**eqn13.26) as

$$e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)+m\mathbf{v}\nu_{coll}=0 \tag{13.29}$$

Retain the definition $\mathbf{j}=ne\mathbf{v}$; so that

$$\frac{m\nu_{coll}}{ne}\mathbf{j}+\frac{1}{nc}\mathbf{j}\times\mathbf{B}=-e\mathbf{E} \tag{13.30}$$

Because we have cross products, things get complicated .. choose the form,

$$\mathbf{j}=\sigma_o\mathbf{E}_{\|}+\sigma_{\perp}\mathbf{E}_{\perp}+\sigma_H\hat{\mathbf{n}}\times\mathbf{E} \tag{13.31}$$

Put this into (13.30), do the algebra and collect the terms[5] We find,

$$\begin{aligned}\sigma_o&=\frac{ne^2}{m\nu_{coll}}\\ \sigma_{\perp}&=\sigma_o\frac{\nu_{coll}^2}{\nu_{coll}^2+\Omega^2}\\ \sigma_H&=\sigma_o\frac{\nu_{coll}\Omega}{\nu_{coll}^2+\Omega^2}\end{aligned} \tag{13.32}$$

where Ω is the gyrofrequency, as usual. These are the collisional, Pederson and Hall conductivities.

[4] Quick physics here: a single particle undergoes gyromotion around the local magnetic field, at a frequency $\Omega=eB/mc$. If there is also a perpendicular $\mathbf{E}$ field, the center of the gyro-orbit shifts during one orbit. Following the particle's trajectory, it's easy to show that the orbit-center moves at a steady drift speed, $\propto\mathbf{E}\times\mathbf{B}$.

[5] Remember the definitions: $\omega_p^2=4\pi ne^2/m$, and $\Omega=eB/mc$; that's how the frequencies ω_p and Ω come into the expression.

• **microscopic**. Alternatively, consider single particle motion. The particles undergo gyromotion around $\mathbf{B}$; they undergo an $\mathbf{E}\times\mathbf{B}$ drift; and they suffer collisions which disrupt these ordered motions. To illustrate (I take this from Park), let $\mathbf{B}=(0,0,B)$ and $\mathbf{E}=(E_x,0,E_z)$. The equations of motion, in the absence of collisions, are

$$\begin{aligned}m\frac{dv_x}{dt}&=qE_x+qBv_y\\ m\frac{dv_y}{dt}&=qBv_x\\ m\frac{dv_x}{dt}&=qE_z\end{aligned}$$

with solutions

$$\begin{aligned}v_x&=v_{xo}\cos\Omega t+\frac{v_{yo}+qE_x}{qB}\sin\Omega t\\ v_y&=\frac{v_{yo}+qE_x}{qB}\cos\Omega t-v_{xo}\sin\Omega t-\frac{qE_x}{qB}\\ v_z&=v_{zo}+\frac{qE_z}{m}t\end{aligned} \tag{13.33}$$

(This shows the combination of gyromotion and $\mathbf{E}\times\mathbf{B}$drift). Now, include collisions. Let ν_{coll} be the collision frequency, and assume collisions occur randomly in time. The probability of a collision is $e^{-\nu_{coll}t}$n and the probability that a particle will escape a collision in time $t, t+dt$ is given by $\nu_{coll}e^{-\nu_{coll}t}dt$. Inbetween collisions the trajectories of (13.33) are followed. Thus, the drift speeds are

$$\begin{aligned}\langle v_x\rangle&=A\int\nu_{coll}e^{-\nu_{coll}t}(E_x/B)\sin\Omega t dt\\ &=\frac{E_x}{B}\frac{\nu_{coll}\Omega}{\nu_{coll}^2+\Omega^2}\\ \langle v_y\rangle&=A\int\nu_{coll}e^{-\nu_{coll}t}(E_x/B)(\cos\Omega t-1)dt\\ &=-\frac{E_x}{B}\frac{\Omega^2}{\nu_{coll}^2+\Omega^2}\\ \langle v_z\rangle&=A\int\nu_{coll}e^{-\nu_{coll}t}(E_x/B)\sin\Omega t dt\\ &=\frac{E_z}{B\nu_{coll}}\end{aligned} \tag{13.34}$$

if the normalizing factor $A^{-1}=\int\nu_{coll}e^{-\nu_{coll}t}dt$. Finally, use this in the definition of current density, $\mathbf{j}=$

$nq\langle \mathbf{v} \rangle$. We get

$$
\begin{aligned}
j_x &= \frac{ne}{B}\frac{\Omega\nu_{coll}}{\nu^2+\omega^2}E_x \\
j_y &= -\frac{ne}{B}\frac{\Omega^2}{\nu^2+\omega^2}E_x \\
j_z &= \frac{ne}{B}\frac{\Omega}{\nu_{coll}}E_z
\end{aligned}
\tag{13.35}
$$

Compare this to (13.32); you can see we get the same answers.

F. Appendix II: Do It in SI

Both SI and cgs are used in the literature, which makes it confusing to go between sources. For reference, I store the main equations in SI.

1. MAXWELL IN SI:

$$
\begin{aligned}
\nabla \times \mathbf{B} &= \mu_o \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_o}\rho_q
\end{aligned}
\tag{13.36}
$$

2. INDUCTION EQUATION

If we eliminate $\mathbf{E}$ from Maxwell # 1, Maxwell # 3 and (13.3), we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{13.37}$$

where, in SI, $\eta = 1/\mu_o\sigma$ is the *magnetic diffusivity*; note that it differs from the *electrical resistivity*, which is $1/\sigma$.

Note of caution: just to confuse us all, the usual cgs definition is $\eta = 1/\sigma$, and lightspeed terms are carried through the equations explicitly. In addition, some authors working in SI also take $\eta = 1/\sigma$. Sigh . . . be sure to check the definitions in each different book you use.]

3. FORCE EQUATION, MAGNETIC TENSION AND PRESSURE

The force equation is

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{F} \tag{13.38}$$

with the separation of terms, as

$$\mathbf{j} \times \mathbf{B} = -\frac{1}{2\mu_o}\nabla B^2 + \frac{1}{\mu_o}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{13.39}$$

As disscussed in §13.C, the first term represents the gradient of a scalar pressure, which in SI is$p_B = B^2/2\mu_o$. The second term is non-zero of $\mathbf{B}$ varies parallel to its length, and represents magnetic tension.

4. ENERGY EQUATION

After dropping the charge-based $\mathbf{E}$, but keeping the Poynting flux explicitly:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \frac{1}{\gamma-1}p + \frac{1}{2\mu_o}B^2\right] + \nabla\cdot\left[\frac{1}{2}\rho v^2\mathbf{v} + \frac{\gamma}{\gamma-1}p\mathbf{v} + \frac{1}{\mu_o}\mathbf{E}\times\mathbf{B}\right] = -\frac{1}{\sigma}j^2 \tag{13.40}$$

and that's the end of this chapter, folks.