14. SIMPLE MHD EQUILIBRIA

How many ways can you construct a steady magnetic field? When you have done that, can it confine a plasma? There are several common applications.

A. Potential Fields

Now and then we can simplify by considering static magnetic fields in vacuum – that is fields arising from a current which is confined to some finite spatial region. A typical example is the magnetic field of the sun, above the solar surface.

Go back to Maxwell: in a current-free region, the magnetic field satisfies $\nabla \times \mathbf{B} = 0$. That means it can be found from a scalar potential, Φ_m. Because $\nabla \cdot \mathbf{B} = 0$ always, the potential (and field) satisfy

$$\nabla^2 \Phi_m = 0 \, ; \quad \mathbf{B} = -\nabla \Phi_m \tag{14.1}$$

But this is Laplace's equation, and has well-known solutions, usually expressed as series expansions. For instance, go back to your favorite E&M book. In spherical geometry the general solution is straightforward (if long), and can be expresed in terms of spherical harmonics:

$$\Phi_m(R,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[a_{lm} r^l + b_{lm} r^{-(l+1)} \right] P_l^m(\cos\theta) e^{im\phi} \tag{14.2}$$

and for axisymmetry, this simplifies to

$$\Phi_m = \sum_l \left[a_l r^l + b_l r^{-(l+1)} \right] P_l(\cos\theta) \tag{14.3}$$

where P_l is the Legendre polynomial. The coefficients a_l, b_l or a_{lm}, b_{lm}, are determined by the boundary conditions. Working in *cylindrical geometry*, with coordinates (r, ϕ, z), is a bit less general, because we have to choose particular boundary conditions. One choice of solution is

$$\Phi_m = \sum_{m,n} \left[c_n J_n(k_{mn} r) + d_n Y_n(k_{mn} r) \right] e^{in\phi \pm k_{mn} z} \tag{14.4}$$

where J_n and Y_n are Bessel functions, and the eigenvalues k_{mn} are chosen by looking at zeros of the Bessels (for instance see Jackson's E&M).

B. Plasma Confinement

Plasma confinement is the fundamental problem for laboratory plasmas. There is no real parallel in fluid dynamics, as the fluid equations do not provide for intrinsic static equilibria (unless a gravitational field is included). The plasma equations do, in principle, provide for self-confinement: if the plasma pressure can just balance the Lorentz forces from the fields. Most commonly, flows are ignored, as is resistivity. The general condition for equilibrium is, then,

$$\frac{\mathbf{j}}{c} \times \mathbf{B} = \nabla p \tag{14.5}$$

This is, of course, can be solved, subject to the constraints

$$\nabla \cdot \mathbf{B} = 0 \, ; \quad \nabla \cdot \mathbf{j} = 0 \, ; \quad \mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B} \tag{14.6}$$

(The second relation holds in steady state, right?). A system which satisfies (14.5) also obeys

$$\mathbf{j} \cdot \nabla p = 0 \, ; \quad \mathbf{B} \cdot \nabla p = 0 \tag{14.7}$$

That is, constant-pressure surfaces are also "magnetic surfaces" and "current surfaces": $\mathbf{B}$ and $\mathbf{j}$ lines lie in constant-p surfaces.

Now... (14.5), with its auxilaries (14.6) and (14.7), is "all" that is needed for laboratory confinement. We just have to solve it, and then test for stability. In these notes I confine myself to infinitely long plasmas in cylindrical geometry. The basic equation, (14.5), becomes

$$\frac{dp}{dr} + \frac{d}{dr} \left(\frac{B_\phi^2 + B_z^2}{8\pi} \right) + \frac{B_\phi^2}{4\pi r} = 0 \tag{14.8}$$

There are several common applications in the literature.

1. THETA PINCH

The simplest is a field which is everywhere parallel to the axis of the cylinder. For instance (following Woods), set up azimuthal currents in the wall of a cylinder which contains plasma; opposing currents are induced in the plasma, and the resulting axial magnetic field, B_z, can't easily penetrate the plasma. Instead, the magnetic force compresses or "pinches" the plasma, until a pressure balance is reached.

To quantify this: if $\mathbf{B} = (0, 0, B_z)$, then confinement must have B_z rising *outside* of the plasma. The equation is simple:

$$\frac{d}{dr} \left(p + \frac{B^2}{8\pi} \right) = 0 \tag{14.9}$$

so that we have a simple, radial pressure balance. For instance, one possible equilibrium (illustrated in the figure) is

$$p(r) = p_o e^{-r^2/a^2}$$
$$B_z(r) = B_o \left[1 - \beta_o e^{-r^2/a^2}\right] \qquad (14.10)$$

with $\beta_o = 8\pi p_o / B_o^2$.

Such a system is effectively confined by the box (which carries the ϕ-currents). We will see that it is stable; however losses out the ends are severe, so that it isn't particularly useful for laboratory confinement.

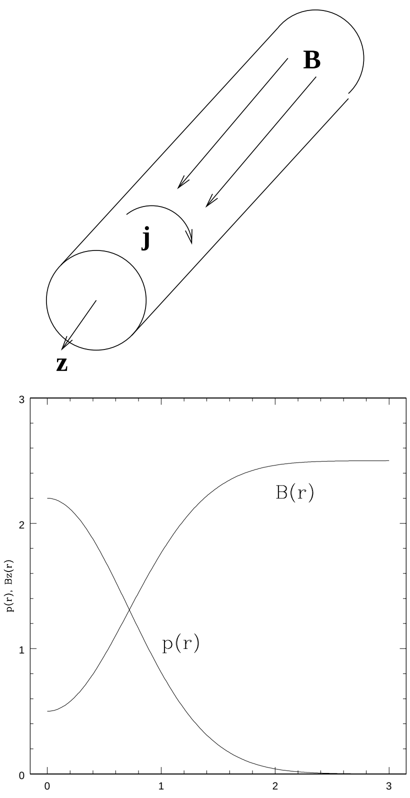

Figure 14.1. Above, the geometry of a linear theta pinch; note the B field is axial and the current azimuthal. Below, *qualitative* equilibrium profiles for a theta pinch. Following Freidberg figure 15.1, 15.2

2. BENNET PINCH OR Z PINCH

The other limit is a purely azimuthal field: $\mathbf{B} = (0, B_\phi, 0)$, so that magnetic tension that confines the plasma. The most interesting case is that in which the plasma confines itself by carrying just the right net current I_o. The basic relation is

$$\frac{d}{dr}\left(p + \frac{B_\phi^2}{8\pi}\right) = -\frac{B_\phi^2}{4\pi r} \qquad (14.11)$$

One possible equilibrium (illustrated in the figure) is

$$B_\phi(r) = \frac{B_o r}{1 + r^2/a^2} \qquad (14.12)$$

(exercise for the student: what are the corresponding pressure and current density profiles?) This type of pinch can be self-confining. Note that the current within radius r is $I(r) = \int_0^r 2\pi j_z r dr$, and from Maxwell the B field is $B_\phi(r) = 2I(r)/rc$. Using these and the pressure balance condition (14.11),

$$\int_0^a p r dr = \frac{1}{4\pi c^2} I_a^2 \qquad (14.13)$$

where I_a is the current in the entire pinch (out to radius a), and we've assumed $p(a) = 0$. (To the student: can you derive this?). Thus, the plasma can self-confine if it carries the right current. This type of pinch is attractive, in that particles don't escape out the ends, and the current is carried by the plasma itself. However, this configuration turns out to be seriously unstable, thus is also of little practical interest.

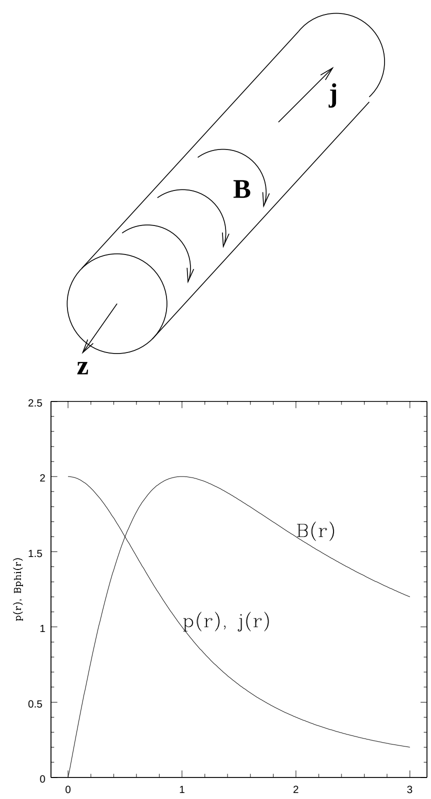

Figure 14.2. Above, the geometry of a linear pinch: the field is azimuthal and the current is axial. Below, *qualitative* solutions for a linear pinch. Following Freidberg figures 15.4, 15.5

3. GENERAL SCREW PINCH

A combination of these two cases is clearly possible, in which both B_z and B_ϕ are non-zero. The general equilibrium is, then, (14.8). One example, which may be considered in more detail in the homework, comes from Freidberg: consider

$$B_\phi = \frac{2I}{c}\frac{r}{r^2+a^2} \tag{14.14}$$

where I is the plasma current and a is a scale length. Let the pressure and B_z field be related by

$$B_z^2(r) = B_o^2 - \lambda p(r) \tag{14.15}$$

where B_o is the externally applied field and λ is a constant, determined experimentally.

A mixed pinch of this type turns out to be more stable than the simple Z pinch, due to its axial field. However it retains the problem of losses out the ends. The next step might be to bend the ends of the screw pinch together, to form a toroid. The simplest form of that is a *tokamak*; many, many trees have been lost to suppor the books and papers analyzing this geometry. It turns out still to suffer serious instability problems. You can go one step further by twisting the cylinder, about its axis, before connecting the ends...that gives you a *stellarator*. I have no intention of pursuing the specialized geometry that these machines require, in these notes.

C. Force-Free Fields

Another track considers solutions to the MHD force equation when both gravity and plasma pressure are negligible. In this case, we must have

$$\mathbf{j}\times\mathbf{B} = 0\,; \qquad \nabla\times\mathbf{B} = \alpha\mathbf{B} \tag{14.16}$$

where α is, thus far, some general function. These are *force-free* fields; they are discussed in lab plasma literature and occasionally in astrophysics. Now, note that we cannot say there is no plasma pressure; there must be a net current, by definition, so there must be some matter. Rather, we are saying the plasma pressure is small – very small – compared to the magnetic forces. This approximation holds well for some laboratory plasma configurations (see “Taylor relaxation”, later in the course). Astrophysically, it is often applied in the solar corona, where $p_b \gg p_{gas}$; and can be applied to other idealized, strongly magnetized systems (such as pulsar magnetospheres and, possibly, radio jets).

The simplest approach here is to pick a simple form for α, usually constant, and look for analytic solutions. Note one simple consequence: taking the divergeance of (14.16), we have

$$\nabla\cdot(\alpha\mathbf{B}) = (\mathbf{B}\cdot\nabla)\alpha = 0 \tag{14.17}$$

so that α must be constant along field lines; that is, $\mathbf{B}$ lies on surfaces of constant α; these are magnetic flux surfaces. If we take $\alpha =$ constant everywhere, in fact, (14.16) becomes

$$\left(\nabla^2+\alpha^2\right)\mathbf{B} = 0 \tag{14.18}$$

Now: the mathematically astute among you will recognized immediately that this is the vector Helmholtz equation. (The rest of you – like me – can refer, for instance, to Morse & Feshbach chapter 13). It has well-understood, if complicated, solutions. Here, I quote a couple of useful solutions without deriving them.

1. CYLINDRICAL GEOMETRY

The system becomes (following Priest)

$$\begin{aligned} \frac{d}{dr}\left(\frac{B_\phi^2+B_z^2}{8\pi}\right) + \frac{B_\phi^2}{4\pi r} &= 0 \\ -\frac{dB_z}{dr} &= \alpha B_\phi \end{aligned} \tag{14.19}$$

But this has Bessel function solutions:

$$B_\phi(r) = B_o J_1(\alpha r)\,; \quad B_z(r) = B_o J_o(\alpha r) \tag{14.20}$$

as originally found by Lüst & Schlüter (1954). These solutions describe helical fields, lying on coaxial cylindrical magnetic surfaces; the pitch of the helices increases going away from the axis.

2. SPHERICAL GEOMETRY

Chandrasekhar & Kendall (1955) found the general solution, in terms of spherical harmonics and spherical Bessel functions. The axisymmetric version of this ($m = 0$ in the general Y_{lm} term) is

$$\psi = \sum_n A_n r^{-1/2} J_{n+(1/2)}(\alpha r) P_n(\cos\theta) \tag{14.21}$$

which yields the following

$$
\begin{aligned}
B_r &= \sum_n C_n n(n+1) r^{-3/2} J_{n+(1/2)}(\alpha r) P_n(\cos\theta) \\
B_\theta &= \sum_n C_n \left[-n r^{-3/2} J_{n+(1/2)}(\alpha r) + \alpha r^{-1/2} J_{n-(1/2)}(\alpha r) \right] \frac{dP_n}{d\theta} \\
B_\phi &= -\sum_n C_n \alpha r^{-1/2} J_{n+(1/2)}(\alpha r) \frac{dP_n}{d\theta}
\end{aligned}
\tag{14.22}
$$

These solutions describe nested, toroidal flux surfaces; the field lines are again helices on the flux surfaces.

3. NON-LINEAR FIELDS

A third interesting case has uniformly twisted solutions, and does not assume $\alpha = constant$. In cylindrical geometry, a field line is traced in (ϕ, z) by

$$\frac{r d\phi}{B_\phi} = \frac{dz}{B_z} \tag{14.23}$$

Thus, the twist of a field line between 0 and z is

$$\varphi(z) = \int d\phi = \int_0^z \frac{B_\phi}{r B_z} dz \tag{14.24}$$

Now, a field whose line-twist is independent of r must satisfy

$$\frac{B_\phi}{B_z} = br \tag{14.25}$$

Requiring that this field also be force-free, that is has $\mathbf{j} \times \mathbf{B} = 0$, gives

$$B_z \frac{dB_z}{dr} + \frac{B_\phi}{r} \frac{d}{dr}(r B_\phi) = 0 \tag{14.26}$$

Combining this with $b =$ constant gives this solution

$$B_z = \frac{B_o}{1 + b^2 r^2}\ ; \qquad B_{\phi(r)} = \frac{B_o b r}{1 + b^2 r^2} \tag{14.27}$$

This has the property that field lines at different radii are twisted through the same angle; so that the whole tube is twisted like a solid body.

4. BOUNDARIES

These three solutions illustrate a general result: there are no force-free fields for which $\mathbf{j} \propto \nabla \times \mathbf{B}$ is confined to a finite volume (as we would expect for a finite source region), and for which $\mathbf{B}$ is $O(r^{-3})$ at infinity. Thus, we need to impose physical boundaries, or other conditions, to keep things well behaved. For cylindrical solutions, a physical boundary at some outer radius makes sense (say from external pressure or a wall). For the spherical solution just discussed, we can also impose a wall (as in the lab), or can also assume the α constant is non-zero only inside of some R. This requires that the usual magnetic boundary conditions be met across $r = R$, and keeps the currents (sources) confined to a finite region.

D. Gravitational Equilibrium I

In Chapter 6 we addressed the question of hydrostatic equilibrium: the static support of a fluid, by its own pressure, in a gravitational field. When we add a magnetic field, the static balance becomes much more complex:

$$\nabla p = \rho \mathbf{g} + \frac{\mathbf{j}}{c} \times \mathbf{B} \tag{14.28}$$

Recall the gravitational acceleration can be written in terms of a potential, $\mathbf{g} = -\nabla \Phi_g$. The three-dimensional nature of the Lorentz force, $\mathbf{j} \times \mathbf{B}$, makes this a much more complicated problem if we treat it fully generally. Let's avoid that and find some simple approaches.

1. PLANAR GEOMETRY: THE GALAXY

Let's start with a nice and simple case, originally due to Parker.

Consider a simple planar system, such as the plane of the galaxy. Let the magnetic field be horizontal, and fully mixed with the gas. Picking $\hat{\mathbf{z}}$ as the vertical direction, that means we take $\mathbf{B} = B_y(z)\hat{\mathbf{y}}$; and describe the gas by density $\rho(z)$, pressure and sound speed $p(z) = c_s^2 \rho(z)$, and gravitational potential $\Phi_g(z)$.

The simplest assumption we can make is that the ratio of gas to magnetic pressure is constant at all altitudes: $B^2/8\pi p = \alpha_o$ = constant. Magnetostatic bal-

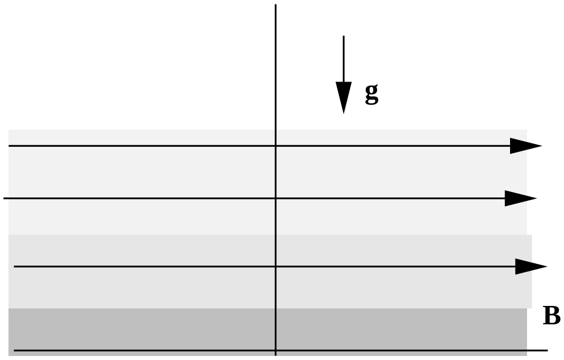

Figure 14.3. A cartoon of the vertical structure of the magnetic field and gas structure in the disk of the galaxy. The grey scale is meant to represent the local density, which decays vertically (presumably as an exponential if g is constant). from Shu figure 23.3.

ance, then, can be written

$$(1+\alpha_0)\frac{dp}{dz} = -\frac{p}{c_s^2}\frac{d\Phi_g}{dz} \tag{14.29}$$

which has the solution,

$$\begin{aligned} p(z) &= p(0)\exp\left[-\frac{\Phi_g(z)}{(1+\alpha_o)c_s^2}\right] \\ B_y(z) &= B_y(0)\exp\left[-\frac{\Phi_g(z)}{2(1+\alpha_o)c_s^2}\right] \end{aligned} \tag{14.30}$$

Thus, the field as well as the gas must be concentrated toward the central plane of the galaxy. If we take a constant gravitational acceleration, then $\Phi_g(z) = g_o|z|$, and we recover the expected exponential dependence of the pressure (and the field).

Looking to useful tools ahead, we can also find the vector potential of this field. Recall that $\mathbf{B} = \nabla\times\mathbf{A}$ in general. In this geometry, $\mathbf{A}$ only has a component in the x direction, $\mathbf{A} = A_x(z)\hat{\mathbf{x}}$, and $B_y(z) = dA_x/dz$. In this simple geometry, it's easy to "invert" the solution for $B_y(z)$ and show that

$$\begin{aligned} A_x(z) = -\mathrm{sgn}(z)\frac{2}{g_o}(1+\alpha_o)^2 B_y(0) \\ \times \exp\left[-\frac{g_o|z|}{2(1+\alpha_o)c_s^2}\right] \end{aligned} \tag{14.31}$$

Note that the magnetic field lines lie in surfaces of constant A_x.

E. Magnetic Bouyancy

The solution above has a difficulty: it is unstable. Magnetic fields tend to be buoyant, thus want to rise against gravity. There are several approaches to this situation, depending on the problem one wants to solve. We'll work through two of them.

1. "CONVECTIVE" INSTABILITY

Think back to the hydrostatic atmospheres in chapter 6. There, we considered a small vertical displacement of a "blob" (that is, gas parcel), and compared its density after displacement to the local, external density. We found that a hydrostatic atmosphere is convectively unstable – the blob will keep rising – if the atmospheric temperature decreases more slowly with height than a fiducial value (the adiabatic gradient).

Now, let's do an analogous calculation with a magnetized atmosphere. To simplify we choose an isothermal atmosphere; that would be stable according to our analysis in chapter 6, but it can be unstable here. Assume the magnetic field lines are horizontal; displace a magnetized parcel vertically by some δz, *without bending the field lines*. [1] The parcel starts at the conditions of the ambient atmosphere: $\rho_{in} = \rho_o$ and $B_{in} = B_o$. Let the parcel density and field change by $\delta\rho_{in}$ and δB_{in}. By flux freezing and mass conservation, with this geometry, we know *for the parcel* that

$$\delta B_{in}/B_o = \delta\rho_{in}/\rho_o \tag{14.32}$$

Now, we again assume the parcel remains in pressure balance with its surroundings: $p_{in} + B_{in}^2/8\pi = p_o^2 + B_o^2/8\pi$. This translates to

$$\delta p_{in} + \frac{B_o}{4\pi}\delta B_{in} = \delta p_o + \frac{B_o}{4\pi}\delta B_o \tag{14.33}$$

But now, if we assume the temperature doesn' change, we get two results.

First, we know that the parcel will continue to rise if $\delta\rho_{in} < \delta\rho_o$. As long as T_{in} is not too different from T_{out}, we see from (14.33), that $\delta B_{in} > \delta B_o$ is required for instability. That makes sense: a larger B in the risen parcel requires a smaller ρ, to stay in pressure balance; thus, bouyancy. It's more useful, however, to get our instability condition in terms of the atmospheric structure. Using (14.32) and (14.33), again with the condition that $\delta\rho_{in} < \delta\rho_o$, we get the condition for bouyant instability:

$$\frac{\delta B_o}{B_o} < \frac{\delta\rho_o}{\rho_o}\ ; \qquad \frac{d}{dz}\left(\frac{B_o}{\rho_o}\right) < 0 \tag{14.34}$$

Thus: an isothermal atmosphere can be unstable if the magnetic field strength falls off faster than the density does.

[1] So this is really displacing a "flux tube", a cylindrical object containing field and plasma.

This analysis can be extended by considering adiabatic changes of the parcel density and temperature, as in chapter 6; we may explore this in class or in the homework.

2. SCALE OF UNSTABLE PERTURBATIONS: PARKER INSTABILITY

Now, let the magnetic structure bend as it rises. The gas is free to flow along the lines, and thus will accumulate in the "valleys"; the "tops" of the bent field lines will thus be lighter than the surrounding gas, and will rise due to bouyant forces. This instability turns out to be important for long wavelenghts: only large perturbations are unstable.

To see this, first consider an isolated flux tube, such as might give rise to a sunspot. Let the external gas have a density scale height $H = k_B T/mg$ (refer back to chapter 6 for hydrostatic equilibrium with no magnetic field). If the magnetic field is confined in the flux tube, and the intial state is in pressure balance, we have the internal/external balance,

$$p_i + \frac{B^2}{8\pi} = p_e \tag{14.35}$$

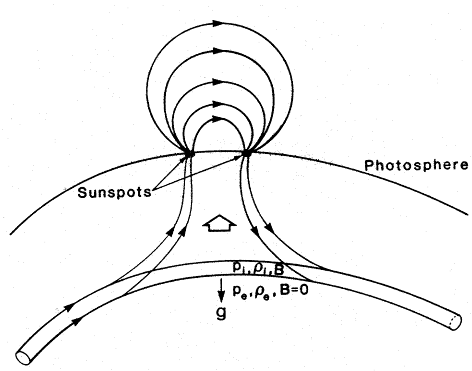

Figure 14.4. The geometry of a sub-surface flux tube before it erupts from the sun due to bouyancy, and its possible post-eruption state; from Tajima &Shibata figure 3.17.

Assuming the gas inside is at the same temperature, it must be at lower density than the outside. Thus leads to a bouyancy force,

$$F_{buoy} = g\left(\rho_e - \rho_i\right) = g\Delta\rho = \frac{B^2}{8\pi H} \tag{14.36}$$

Say, now, that the flux tube is bent upwards, locally, with a radius of curvature λ. If λ is short, magnetic tension will pull the tube back towards its initial position, giving a stable system. If, however, λ is long, the buoyancy force will overcome the tension, leading to instability. Comparing these two forces, we find instability occurs if

$$g\Delta\rho > \frac{B^2}{4\pi\lambda}\,; \qquad \lambda > 2H \tag{14.37}$$

We can make the same argument for the magnetic bouyant (in)stability of a flat flux sheet (as in the planar model of the galaxy, above). Our static solution gives a new scale height:

$$\frac{p}{p_o} = \frac{\rho}{\rho_o} = \frac{B^2}{B_o^2} = e^{-z/\Lambda} \tag{14.38}$$

where $\Lambda = \left(c_s^2 + v_A^2/2\right)/g$, in direct extension of the nonmagnetized scale height from above (check back to Chapter 6). Let the flux sheet be raised some small Δz, with the gas sliding down into the "valleys" again. If the perturbation has horizontal scale λ, its effective curvature radius is, from simple geometry,

$$r \simeq \frac{\lambda^2}{16}\frac{2}{\Delta z}$$

The density at the top of the perturbation is now (take "o" to be at the height of the undisturbed sheet)

$$\rho_i(\Delta z) \simeq \rho_o e^{-\Delta z/H} \simeq \rho_o\left(1 - \frac{\Delta z}{H}\right) \tag{14.39}$$

(Note there is no magnetic pressure supporting the gas inside the bent flux sheet). The change in external density does know about magnetic pressure however:

$$\rho_3(\Delta z) \simeq \rho_o e^{-\Delta z/\Lambda} \simeq \rho_o\left(1 - \frac{\Delta z}{\Lambda}\right) \tag{14.40}$$

If we again compare bouyancy to the restoring force of magnetic tension, we find the instability condition for this case:

$$\lambda^2 > \frac{16\Lambda^2}{\left(1 + 1/\beta\right)^2} \tag{14.41}$$

if $\beta = 8\pi p/B^2$ is the ratio of gas to magnetic pressure.

References

The general cylindrical MHD equilibria are discussed in several books; Priest and Bateman are good sources. Moffatt discusses the force-free solutions. I took the Magnetic Buoyancy discussion from Shu.