# 15. NOT-SO-SIMPLE EQUILIBRIA

Most magnetostatic systems are a bit more complex than the simple ones in Chapter 14. In this chapter we'll look at two important methods. First we'll look at ways to describe MHD equilibria, by reducing vector DE's to scalar DE's (through *flux functions*, described below). Then we'll look at a process by which an MHD system, left to itself, will "relax" to its own equilibrium (this is called Taylor relaxation).

## A. Flux functions I: Gravitational Equilibrium Revisited

To start, think back to the last chapter, which we ended with simple, planar equilibria. What if we can't assume the field is planar? Our basic magnetohydrostatic equation,

$$-\nabla p + \rho \mathbf{g} + \frac{1}{c}\mathbf{j} \times \mathbf{B} = 0 \tag{15.1}$$

is now a more complicated vector equation, and hard to solve. becomes harder to solve. How can we proceed to search for an equilibrium (stable or unstable)? A common procedure introduces what is called a *flux function* – that's a scalar function that can be used to find the vector $\mathbf{B}$ field.[1]

Before we start here, you should check back to the *stream function*, $\psi$, that we introduced in chapter 3. Some points about that $\psi$ are good analogs for what we're doing here, as follows.

- The stream function is a scalar that can be used to find the vector $\mathbf{v}$ field.
- Solutions for $\psi(\mathbf{r})$ are generally found by solving some sort of DE.
- The stream function $\psi$ "labels" streamlines. From this, it follows that $\delta\psi = \psi_B - \psi_A$ (the difference in $\psi$ values on two streamlines A and B), is proportional to the fluid flow between those streamlines. (You can work this out if you don't believe me.)

So, now let's do some algebra. We're working in Cartesian geometry here.

### 1. DEFINE THE FLUX FUNCTIONS

To start, pick a two-dimensional problem: let the field $\mathbf{B} = \mathbf{B}(x,z) = (B_x, 0, B_z)$. A common approach allows us to reduce the vector equation (15.1) to a scalar equation. To do so we must introduce flux functions. In this section we'll work in Cartesian coordinates; later we'll use a similar approach in cylindrical.

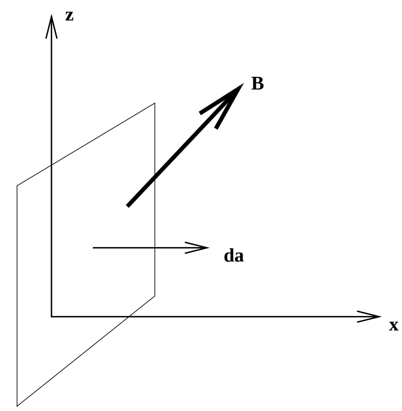


**Figure 15.1.** Geometry for Cartesian problem. The magnetic flux through the square is $\Phi_B = \int \mathbf{B}\cdot d\mathbf{a}$. Using our definition (15.2), and working per unit length in the $y$ direction, $\Phi_B = \int B_x da = \int \frac{\partial \Psi}{\partial z} dz = \int d\Psi$. See the text for more details.

If the $B$ field has only two components, we may choose its vector potential to have only a single component, $\mathbf{A} = (0, \Psi, 0)$. We then have

$$B_x = \frac{\partial \Psi}{\partial z}\ ;\ \ B_z = -\frac{\partial \Psi}{\partial x}\ ;\ \ \mathbf{B} = \nabla\Psi \times \hat{\mathbf{y}} \tag{15.2}$$

It's easy to show that the magnetic field lines lie on surfaces of constant $A$. We can see this from (15.2), by simply noting that $\mathbf{B}$ is perpendicular to the gradient of $A$; thus it must lie in constant-$\Psi$ surfaces. Alternatively, we can consider the differential of $\Psi$ along a field line:

$$d\Psi = \frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial z}dz = -B_z dx + B_x dz \tag{15.3}$$

But, a field line obeys

$$\frac{dx}{B_x} = \frac{dz}{B_z} \tag{15.4}$$

Thus, (15.3) shows that $d\Psi = 0$ along a field line. Thus, the scalar function $\Psi$ labels magnetic surfaces (or magnetic field lines in a 2D representation of the geometry).

The function $\Psi$ can also be called a *flux function*. To see why, recall the integral form of the defining equation for $\mathbf{A}$:

$$\Phi_B = \int \mathbf{B}\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l} \tag{15.5}$$

[1] This approach may look intimidating, but people do it because it's almost always easier to solve one scalar equation than three inter-related equations for vector components.

For our geometry, consider the magnetic flux through a surface taken in the $(x, z)$ plane (think of a unit length in $y$ to make a reasonable mental picture, so $da = dydz \to dz$). This is just

$$\Phi_B = \int_o^z \frac{\partial \Psi}{\partial z} dz = \Psi(z) - \Psi(0) \tag{15.6}$$

Thus: if we take $\Psi(0) = 0$, then $\Psi(z)$ measures the magnetic flux contained "within" (below in this case) the surface of constant $\Psi$.

2. APPLY THEM: SOLAR MAGNETIC ARCHES

Now: put this formalism into the magnetostatic equation, (15.1). We note several useful facts.

*Fact 1.* The associated current is easy to find:

$$\mathbf{j} = (c/4\pi)(\nabla^2\Psi)\hat{\mathbf{y}}$$

You can find this in your favorite EM book, or derive it directly from $\nabla \times (\nabla \times \mathbf{A})$.[2] From (15.2) and (15.1), the components of the Lorentz force are

$$\frac{\mathbf{j}}{c} \times \mathbf{B} = \frac{1}{4\pi} \left[ \left(\nabla^2\Psi\right) \frac{\partial \Psi}{\partial x}, 0, \left(\nabla^2\Psi\right) \frac{\partial \Psi}{\partial z} \right] \tag{15.7}$$

which can also be written,

$$\frac{\mathbf{j}}{c} \times \mathbf{B} = \frac{1}{4\pi} \left(\nabla^2\Psi\right) \nabla\Psi \tag{15.8}$$

Details here: The $\nabla^2\Psi$ term is a scalar function, which multiplies the vector $\nabla\Psi$. We have taken an important step: we have reduced our system to one scalar unknown, $\Psi$, rather than the vector unknowns $\mathbf{B}$ and $\mathbf{j}$.

*Fact 2.* What is the pressure distribution? Consider some field line, locally $\mathbf{B}$, at a direction $\theta$ to the vertical. The component of (15.1) parallel to $\mathbf{B}$ is

$$0 = -\frac{dp}{ds} - \rho g \cos\theta; \qquad \frac{dp}{dz} = -\rho g \tag{15.9}$$

if $ds = dz/\cos\theta$ is the differential distance along the field line. The second equality just converts from $ds$ to $dz$, and shows that *normal hydrostatic equilibrium obtains along the field line*. This can be written,

$$p(z) = p_o(\Psi) e^{-\int_o^z dz/H(z)} \tag{15.10}$$

if $H(z) = k_B T(z)/mg$ is the local scale height.[3] I have noted the explicit dependence of $p_o$ on the local value of $\Psi$ at the footpoint of the line (for instance the surface of the sun). This result also shows us that surfaces of constant pressure *nearly* coincide with field lines. For scales $\ll H(z)$, the pressure is nearly constant, and hydrostatic balance becomes $\mathbf{j} \times \mathbf{B} \simeq c\nabla p$. Dotting this with $\mathbf{B}$ or with $\mathbf{j}$ gives

$$\mathbf{B} \cdot \nabla p \simeq 0\,; \qquad \mathbf{j} \cdot \nabla p \simeq 0\,.$$

Thus, the magnetic field and current lie nearly in surfaces of constant pressure. Deviations occur only on scales $\gtrsim H(z)$ (for an example see the homework).

*Fact 3.* we can consider the full force equation as a system to be solved for the function $\Psi(z)$ (and thus the field structure). The gravitational and pressure terms can be combined as

$$\rho\nabla\Phi_g + \nabla p = -e^{-\Phi_g/c_s^2}\nabla q \tag{15.11}$$

where, in the last step, we have introduced the usual gravitational potential, given by $\mathbf{g} = -\nabla\Phi_g$, and have defined the function $q = pe^{\Phi_g/c_s^2}$. Combining this with (15.8), we can write the full force equation, (15.1), as

$$-\frac{1}{4\pi}(\nabla^2\Psi)\nabla\Psi = e^{-\Phi_g/c_s^2}\nabla q \tag{15.12}$$

This balance requires that $\nabla\Psi$ and $\nabla q$ be parallel to one another; so that surfaces of constant $\Psi$ must also be surfaces of constant $q$. It follows that we can pick $q = q(\Psi)$, *i.e.* that $q$ can be written as a function of $\Psi$. Given this, our basic equation turns into

$$\nabla^2\Psi = -4\pi e^{-\Phi_g/c_s^2}\frac{dq}{d\Psi} \tag{15.13}$$

This is an important result: we have reduced the system from a general vector equation, (15.1), to a quasilinear PDE in $\Psi(x, z)$. If we can pick $q(\Psi)$ (for instance, constant or linear or some other simple function), we can find solutions for $\Psi(x, z)$ by standard PDE methods.

An example of this will appear in class or in the homework.

### B. Flux Functions II: Grad-Shafranov Equation

Another version of flux functions is applied to plasma confinement problems which can be reduced to two independent variables (such as an axisymmetric cylinder problem). Again, the general equilibrium condition, in the absence of gravity, is

$$\frac{\mathbf{j}}{c} \times \mathbf{B} = \nabla p \tag{15.14}$$

[2] BIG NOTE: $\nabla^2\Psi = \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial z^2$ is a *scalar*, with simple form in Cartesian.

[3] Compare to the scale height in Chapter 6 for a uniform $T$.

Once again, it follows that $\mathbf{j} \cdot \nabla p = 0$ and $\nabla p \cdot \mathbf{B} = 0$. That is, $\mathbf{B}$ and $\mathbf{j}$ are normal to $\nabla p$; each of $\mathbf{B}$ and $\mathbf{j}$ lie in constant-$p$ surfaces. These are called *magnetic surfaces* or *flux surfaces*; they contain the field lines. Most interesting equilibria consist of sets of nested magnetic surfaces. Once again, we will label these surfaces with flux functions, $\psi$.[4] We will have $\psi =$ constant in a flux surface; and we can write $p = p(\psi)$, so that $p(\psi)$ is constant on the surface given by $\psi =$ constant.

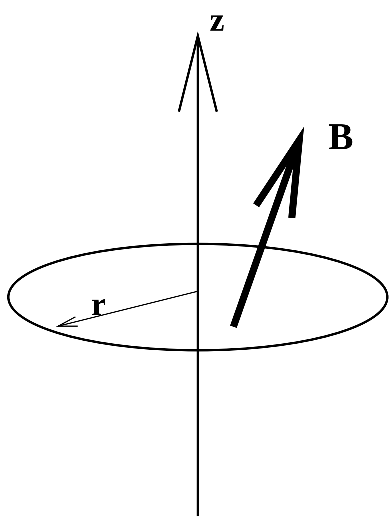


**Figure 15.2.** Geometry for the axisymmetric cylindrical problem. The magnetic flux through the circle, perpendicular to the $z$-axis, is $\Phi_B = \mathbf{B} \cdot d\mathbf{a} = \int B_z 2\pi r dr$. Using (15.16), this becomes $\Phi_B(r) = 2\pi r A_\phi = 2\pi\psi(r, z)$ (you should derive this!). See the text for more details.

### 1. DEFINE THE FLUX FUNCTIONS

We need to introduce two scalar functions. The first is the $\phi$- component of the vector potential, which can be related to $\psi$, the *poloidal flux function*

$$\psi(r, \phi) = A_\phi r \tag{15.15}$$

(Figure 15.2 shows the basic geometry for the simpler, axisymmetric case. As with the Cartesian case, the scalar function $\psi(r)$ labels the flux within a circle of radius $r$. It follows, as before, that the poloidal field components can be written

$$\begin{aligned} \mathbf{B}_{pol} = \nabla \times \left( \hat{\boldsymbol{\phi}} \frac{\psi}{r} \right) \quad \Rightarrow \\ B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \; ; \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \end{aligned} \tag{15.16}$$

[4] Yes, the notation has changed slightly ... its conventional to use (lower case) $\psi$ for flux functions in non-cartesian problems (which tend to be laboratory plasmas, in cylindrical or toroidal geometries).

One function, $\psi$, is enough if the field has no $\phi$ component. For the more general case we need a second scalar function. Consider the total current $J$, within surface $\mathbf{S}$ which is planar and lies perpendicular to the $z$ axis (refer back to Figure 15.2 for such a surface):

$$J = \int_{\mathbf{S}} \mathbf{j} \cdot d\mathbf{a} = \int_0^{2\pi} \int_0^r j_z r d\phi dr \tag{15.17}$$

But now, since $r j_z = \partial(B_\phi r)/4\pi\partial r$ in axisymmetry, we can write

$$J = \frac{c}{2} B_\phi r \tag{15.18}$$

This is our second surface function, $J(\psi)$, the axial current crossing through the surface $\mathbf{S}$. It pairs with $p$ in the analysis, both functions being constant on a $\psi$ surface.

### 2. APPLY THEM: THE G-S EQUATION

To start, we can get the toroidal current density from $j_\phi = (c/4\pi)\left(\nabla^2 \mathbf{A}\right)_\phi$:

$$j_\phi = -\frac{c}{4\pi r}\left[\frac{\partial^2 \psi}{\partial z^2} + r\frac{d}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right] = -\frac{c}{4\pi}\Delta^* \psi \tag{15.19}$$

where we have followed convention and defined

$$\Delta^* \psi = \frac{1}{r}\left[\frac{\partial^2 \psi}{\partial z^2} + r\frac{d}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right] \tag{15.20}$$

to shorten the notation. From this we can write the general $\mathbf{B}$ field in terms of two scalars $J$ and $\psi$:

$$\mathbf{B} = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z}, \frac{2J}{cr}, \frac{1}{r}\frac{\partial \psi}{\partial r}\right) \tag{15.21}$$

Similarly, the total current density is

$$\mathbf{j} = \frac{1}{2\pi}\left(-\frac{1}{r}\frac{\partial J}{\partial z}, \frac{c}{2}\Delta^* \psi, \frac{1}{r}\frac{\partial J}{\partial r}\right) \tag{15.22}$$

These surface functions, $J$ and $\psi$, can be used to reduce the equilibrium condition, (15.14), to one DE rather than three. As we noted abive, because both the pressure $p$ and the axial current $J$ are surface functions, we can write $p = p(\psi)$, and $J = J(\psi)$. Our most useful relation is the $r$-component of (15.11):

$$\frac{\partial p}{\partial r} = \frac{1}{c}\left(j_\phi B_z - j_z B_\phi\right)$$

But now, since $p = p(\psi)$, we can write $\partial p/\partial r = (dp/d\psi)(\partial\psi/\partial r)$. We can then cancel the $\partial\psi/\partial r$

terms, insert the full expression for $j_\phi$, and get one form of the G-S equation:

$$\frac{\partial p}{\partial \psi} + \frac{1}{2\pi r^2 c}\frac{\partial J^2}{\partial \psi} + \frac{1}{4\pi r^2}\left[\frac{\partial^2 \psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right] = 0 \tag{15.23}$$

This is the simplest interesting version of the G-S equation, outside of that in Cartesian coordinates, that I know. (Most are set up for toroidal coordinate systems, to be relevant in the lab, and are truly horrible ..) If $p(\psi)$ and $J(\psi)$ are known functions of $\psi$, then this is a PDE in the scalar flux function, $\psi$. Once we solve this for $\psi$, everything else we need is known (check 15.19, 15.21, 15.22).

### 3. EXAMPLE: SIMPLE PINCHES

To solve the G-S equation, one *chooses* functional forms for $dp/d\psi$, and $dJ/d\psi$, then finds solutions of (15.23).

As a very simple example, let's assume $J(\psi) = 0$, and look for a solution independent of $z$ (think of plasma confinement in a simple pinch geometry). The G-S equation becomes

$$\frac{\partial p}{\partial \psi} + \frac{1}{4\pi r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0 \tag{15.24}$$

But, we know $B_z = (1/r)(d\psi/dr)$; so this equation is just

$$\frac{d}{dr}\left(p + \frac{B_z^2}{8\pi}\right) = 0 \tag{15.25}$$

(does that look familiar?). A slightly more complicated solution is the screw pinch. We again assume the equilibrium is independent of $z$, but retain $J(\psi) \neq 0$. If we multiply the G-S equation by $(4\pi r^2)(d\psi/dr)$, we get

$$\frac{\partial p}{dr} + \frac{B_z}{4\pi}\frac{dB_z}{dr} + \frac{J}{4\pi r^2}\frac{dJ}{dr} = 0 \tag{15.26}$$

But because $J = cB_\phi r/2$, the G-S equation becomes

$$\frac{dp}{dr} + \frac{d}{dr}\left(\frac{B_\phi^2 + B_z^2}{8\pi}\right) + \frac{B_\phi^2}{4\pi r} = 0 \tag{15.27}$$

which recovers our earlier result for cylindrical plasma confinement.

The previous example was overkill; we did not need to go through the G-S equation to recover the result. Most situations are more complex, however, with $z$ as well as $r$ dependence. Searches for a solution start by assuming an analytic form for $p(\psi)$ and $J(\psi)$.[5] Most realistic $p(\psi)$ and $J(\psi)$ choices requires numerical solution of the equation; this is the current approach in the literature. However, some analytic solutions were found early on, and are still useful for gaining insight. You will see an example in class or in the homework.

## C. Helicity and Taylor relaxation

Now, a different track ... return to the force-free fields which we discussed in §14.3. There, we presented some simple solutions; in this section we discuss why they might be interesting.

This subject was motived by laboratory plasmas, which exhibit *self-organization*. These plasmas, in tin cans, tend to evolve spontaneously towards a small number of preferred states, which are independent of the initial conditions of the system. The structure of these preferred states depends only on a few global parameters, such as total magnetic flux, and on the geometry of the system (say, cylindrical or toroidal). Examples are the reversed field pinch (RFP), the spheromak, and the tokamak (each of which have particular stable field configurations, that I'm not going to describe in detail here). The process by which the plasma evolves from an arbitrary initial condition to its long-lived final state is called *plasma relaxation*. Taylor (1974, and later) suggested that this is a variational process – in which the system evolves toward a state of minimum energy, but constrained by one or more invariants. To describe this we need two mathematical tools.

### 1. HELICITY

First, we must define *helicity*. For a magnetic system, the helicity is defined by the integral,

$$\mathcal{H} = \int \mathbf{B}\cdot\mathbf{A}dV \tag{15.28}$$

where the integral is over some relevant closed volume $V$: that of the system, or that of some particular flux surface, depending on the application. Let this volume be bounded by a surface $S$. Helicity can only be uniquely defined in a *closed system*: one in which the magnetic field lines do not cross the surface. To see this, recall that the vector potential $\mathbf{A}$ is only defined up to an arbitrary gradient (the choice of which is called the gauge). Consider some gauge transform;

[5] This sounds arbitrary, yes; but think even of simple solutions we've seen to (15.25) or (15.27). Those problems also start by "assume we know the behavior of at least one of the functions", then proceed to find the behavior of the other(s).

let $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi$, where $\chi$ is some arbitrary function. The effect of this on the helicity is

$$\mathcal{H}' - \mathcal{H} = \int_v \nabla\chi \cdot \mathbf{B} dV = \int_S \chi \mathbf{B} \cdot \hat{\mathbf{n}} dS \quad (15.29)$$

Thus, $\mathcal{H}$ is uniquely defined only if $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$ (that is, if there is no component of $\mathbf{B}$ across the boundary; $\hat{\mathbf{n}}$ is the surface normal vector.)

• How can we understand helicity? Moffatt & Tsinober (1992) give two useful illustrations. For one illustration, consider a circular magnetic flux tube, in which each field line is a circle running parallel to the axis of the tube. The helicity of this state is (clearly?) zero. Let the magnetic flux of this tube be $\Phi_B$. Imagine now that we cut the tube at any section, twist it through an angle $2\pi\gamma$, and reconnect. Each field line in the tube is now a torus knot; and any pair of field lines is now linked. This new system has finite helicity.
• Or, consider a second, more complicated example. Two magnetic flux tubes, topologically linked through each other. The integral in (15.28) simplifies to two line integrals, around the axes of each tube:

$$\mathcal{H} = \kappa_1 \oint_{\mathcal{C}_1} \mathbf{A} \cdot d\mathbf{l} + \kappa_2 \oint_{\mathcal{C}_2} \mathbf{A} \cdot d\mathbf{l}$$

But now, the first integral is equal to the flux of field through a surface spanning loop $\mathcal{C}_1$, that is $\pm\kappa_2$; and similarly for the second integral. Thus, the net helicity is $\mathcal{H} = \pm\kappa_1\kappa_2$.

Having defined helicity, we now need to introduce two important conservation laws. First, helicity is an invariant in an ideal, closed MHD system. Second, the minimum magnetic energy available to a system at a fixed (invariant) value of the helicity corresponds to that of a force-free field. Here are the proofs for each of these.

## 2. Invariance of the helicity

First, we want to establish that $\mathcal{H}$ is an invariant in ideal flow. The general rate of change can be written,

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial t} &= \int_V \left( \mathbf{B} \cdot \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dV \\ &= \int \left[ -\nabla \cdot \left( \mathbf{A} \times \frac{\partial \mathbf{A}}{\partial t} \right) + 2\frac{\partial \mathbf{A}}{\partial t} \cdot \nabla \times \mathbf{A} \right] dV\end{aligned} \quad (15.30)$$

To work with $\partial\mathbf{A}/\partial t$, we need to make a choice of gauge. Recall the general expression,

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi$$

if $\Phi$ is the scalar electric potential. Note, further, that $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ in an ideal system. If we work now in a gauge where $\nabla\Phi = 0$, we can show $\mathcal{H}$ is constant.

• **method 1.** One way to proceed (following Priest) is in terms of conditions on $\mathbf{A}$. In an ideal system, we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})\,; \qquad \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B}\,. \quad (15.31)$$

The first relation in (15.31) comes from the basic induction equation, for an ideal ($\eta \to 0$) system. The second comes from "uncurling" $\mathbf{B}$ in the first, and choosing a gauge with $\Phi = 0$. From this, (15.30) can be written,

$$\frac{\partial \mathcal{H}}{\partial t} = -\int_S \left( \mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \hat{\mathbf{n}} dS + 2\int_V \mathbf{B} \cdot (\mathbf{v} \times \mathbf{B}) dV \quad (15.32)$$

But the second term is clearly zero. Thus we find that $\mathcal{H}$ is invariant in systems for which $\mathbf{A}$ is held constant on the boundary.

• **method 2.** An alternative proof comes from Biskamp, who works with conditions on $\mathbf{B}$ and $\mathbf{v}$. Still using the fact that $\mathbf{B} \cdot \mathbf{E} = 0$, we rewrite (15.30) as

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial t} &= \int_V \mathbf{A} \cdot \nabla \times (\mathbf{v} \times \mathbf{B}) dV \\ &= \int_S [(\mathbf{A} \cdot \mathbf{v})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{v}] \cdot \hat{\mathbf{n}} dS\end{aligned} \quad (15.33)$$

From this last, we find that $\mathcal{H}$ is invariant if $\mathbf{B} \cdot \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} = 0$; that is, if neither $\mathbf{B}$ nor $\mathbf{v}$ connect across the boundary.

Both of these methods show that $\mathcal{H}$ is constant for an ideal, closed system (highly conductive and not connected to the outside universe).

## 3. The minimum energy state

A second important relation is the fact that when the magnetic energy,[6]

$$W = \frac{1}{8\pi} \int B^2 dV\,, \quad (15.34)$$

[6] This is traditional notation ... $W$ instead of $E$ ... sorry, folks – JAE.

is minimized subject to the constraint $\mathcal{H} = $ constant, the resultant field is force free.

We prove this using a Lagrange multiplier technique. That is, we consider small perturbations to a magnetic system, and search for the state which satisfies

$$\delta\left(W - \alpha\mathcal{H}\right) = 0 \tag{15.35}$$

In particular, following Priest, consider small perturbations which ahve $\delta\mathbf{A} = 0$ on $S$ (the surface; again a fixed $\mathbf{A}$ thereon), and $\delta\mathbf{B} = \nabla \times \delta\mathbf{A}$. The change in $W$ is then (linearizing and subtracting $\alpha\mathcal{H}$),

$$8\pi\delta W = \int 2\mathbf{B}\cdot\delta\mathbf{B} - \alpha\left(\delta\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\delta\mathbf{B}\right)dV \tag{15.36}$$

Algebra allows us to rewrite this as

$$8\pi\delta W = \int \nabla\cdot\left(-2\mathbf{B}\times\delta\mathbf{A} + \alpha\mathbf{A}\times\delta\mathbf{A}\right)dV + 2\int\left(\nabla\times\mathbf{B} - \alpha\mathbf{B}\right)\cdot\delta\mathbf{A}dV \tag{15.37}$$

But the first term again can be written as a surface integral, which vanishes due to $\delta\mathbf{A} = 0$ on $S$. We are left, then, with the second term:

$$\delta W = \frac{1}{4\pi}\int\left(\nabla\times\mathbf{B} - \alpha\mathbf{B}\right)\cdot\delta\mathbf{A}dV \tag{15.38}$$

But this says that we have a minimum energy state if the field is force-free:

$$\delta W = 0 \qquad \text{if} \qquad \nabla\times\mathbf{B} = \alpha\mathbf{B} \tag{15.39}$$

Thus: we have shown that the state of minimum energy, energy, given a specified value of helicty, is a force-free state.

## 4. TAYLOR RELAXATION

This result has an interesting application in the lab (and presumably elsewhere). Lab plasmas are observed to “relax” spontaneously. If a plasma starts in some initial state (specified by the experimenters, say), it goes through a turbulent state (which develops due to instabilities in the initial configuration), and ends up in a final, longer-lived configuration. Taylor hypothesized that this corresponds to the plasma dissipating magnetic energy (through self-generated turbulence), while maintaining its initial value of helicity (as determined by the initial configuration). Its final state should then be a force-free state, as in (14.16). This is indeed observed, most famously in an experiment called the Reverse Field Pinch, which has cylindrical geometry and in which the Bessel-function solutions (14.20) are seen. It has also been suggested to occur in solar flares (which involved a sudden, dramatic release of energy), and even in radio jets (where it has been used to predict the magnetic field structure). We should note, however, that applications to astrophysics, while very tempting, are hampered by the lack of rigid boundaries. The derivations above made heavy use of the tin-can boundary that we have in the lab.

How does this work? How can magnetic energy be dissipated while magnetic helicity is not? There is still a fair amount of discussion about this; the sense of the literature is that this works because magnetic helicity decays much more slowly than magnetic energy does. Following Bellan, think about the dimensions of magnetic energy *vs.* magnetic helicity: $W \sim B^2L^3$, while[7] $\mathcal{H} \sim ABL^3 \sim B^3L^4$, where $L$ is the characteristic linear dimension of the system (or the scale of the Fourier component which holds most of the power). Now, let $B$ be decreased (for instance by resistive dissipation); the total energy decays $\propto L^3$, while the helicity decays $\propto L^4$. Thus, the smaller the scale, the bigger is the ratio of energy loss to helicity loss.

We can do this a bit more formally,following Orlanti & Schnack. Magnetic energy decays due to resistivity, as

$$\frac{dW}{dt} = -\eta\int\mathbf{j}^2 dv \tag{15.40}$$

while helicity decays as

$$\frac{d\mathcal{H}}{dt} = -2\eta\int\mathbf{j}\cdot\mathbf{B}dV \tag{15.41}$$

(to get this last, we started with the definition, (15.28), used $d\mathbf{A}/dt = -c\mathbf{E}$, and Ohm's law). But now, if we think in terms of Fourier components, $\mathbf{B} = \sum_k b_k e^{ikr}$ for a set of wavenumbers $k$, then we have

$$\frac{dW}{dt} \rightarrow -\eta\sum_k k^2 b_k^2 \tag{15.42}$$

and

$$\frac{d\mathcal{H}}{dt} \rightarrow -2\eta\sum_k k b_k^2 \tag{15.43}$$

Thus, again, at a given $k$, the energy decay $\propto k^2$, while the helicity decay $\propto k$.

[7] Remember how $\mathbf{A}$ connects to $\mathbf{B}$.

So, finally, to close the arguments: turbulence, and thus turbulent dissipation, is dominated by small scales (large $k$ values). We therefore expect the energy to dissipate more rapidly than the helicity does – thus supporing Taylor's hypothesis (and its apparent validation in the lab).

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**References**

I've taken the flux-function material and solutions from a mixture of Priest, Bateman and the plasma literature. The helicity and relaxation discussion comes partly from me, and partly from references such as Moffatt and Biskamp, as well as the more specialized books by Bellan, and Orlanti & Schnack.