16. MHD EFFECTS IN FLUID FLOWS

We now have the basics: we know, in principle, how B fields interact with fluid flows. They exert magnetic tension and pressure on the fluid; the fluid, in return, modifies and/or determines the local B field, through inductive effects (that is, the $\mathbf{v} \times \mathbf{B}$ EMF creates local currents). In this chapter we visit some applications.

A. Magnetic damping and stirring

Magnetic fields can have some unexpected effects on fluid flow – if you set things up right, they can either accelerate or decelerate the flow. Here are a couple of brief examples; we may discuss them more in class.

1. MAGNETIC DAMPING.

Magnetic fields can decelerate a flow. For a concrete example, let's say you try to drive a flow (maybe a jet) across a pre-existing $\mathbf{B}$ field. Qualitataively, we know that the flow will initially try to "stretch" the field lines; the resultant magnetic tension will resist – and decelerate – the flow. More quantitatively, the EMF caused by the flow, $\mathbf{v} \times \mathbf{B}$, will generate a transverse current **j**; this will give a "backwards" $\mathbf{j} \times \mathbf{B}$ force which will try to decelerate the flow.

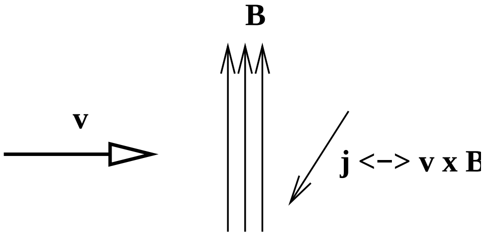

Figure 16.1. Illustrating magnetic damping. Sent a jet, velocity $\mathbf{v}$, into a transverse $\mathbf{B}$ field. The resultant EMF, $\mathbf{v} \times \mathbf{B}$, will drive a current: which way will the Lorentz force act? Will it accelerate or retard the flow?

The energetics are simple. From Ohm's law, $\mathbf{j} = \sigma\left(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}\right)$. If the flow is set up with no free-charge $\mathbf{E}$ (for instance by keeping all boundaries at the same potential), then we find that the rate of work done on the flow, by the Lorentz force, is

$$\frac{1}{c}\mathbf{v} \cdot (\mathbf{j} \times \mathbf{B}) = -\frac{1}{c}\mathbf{j} \cdot (\mathbf{v} \times \mathbf{B}) = -\frac{j^2}{\sigma} \qquad (16.1)$$

Connect this to the force equation (which we dot with $\mathbf{v}$ to get an energy equation):

$$\mathbf{v} \cdot \frac{D}{Dt} = \frac{1}{c}\mathbf{v} \cdot (\mathbf{j} \times \mathbf{B}) - \nabla \cdot (p\mathbf{v}) \qquad (16.2)$$

Integrate this over a finite box (without losses through the sides):

$$\frac{d}{dt}\int \frac{1}{2}\rho \mathbf{v}^2 dV = -\frac{1}{\sigma}\int j^2 dV \qquad (16.3)$$

Thus, we find that the energy lost from the flow, due to the Lorentz force, is just balanced by ohmic losses – which of course appear as heat in the flow: (Compare the more general energy equation, at the end of Chapter 13 – this is a simple application of the same physics.)

2. MAGNETIC STIRRING.

Magnetic fields can also induce motion in a fluid. One simple example is a magnetic field rotated (at some angular frequency Ω) around a fluid that is initially stationary (think about external electromagnets rotating around the cylinder).

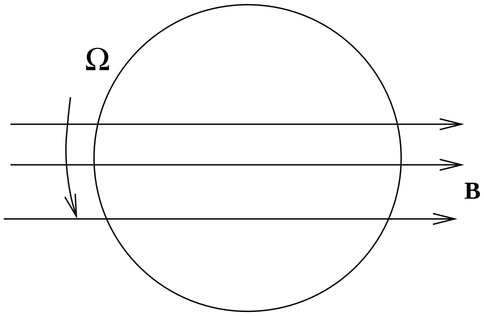

Figure 16.2. Illustrating magnetic damping. A fluid sits in a long cylinder (seen end-on). A magnetic field $\mathbf{B}$ pervades the fluid, and is rotated at some Ω.

How does this cause motion? Think about this in a frame in which the $\mathbf{B}$ field is at rest; in this frame the fluid initially has an azimuthal motion $v_{phi} = \Omega r$. But this gives us an EMF,

$$\frac{\mathbf{v}}{c} \times \mathbf{B} \rightarrow \frac{\Omega r B}{c} \qquad (16.4)$$

This induces a current, as usual (in which direction??), which leads to a Lorentz force

$$\frac{\mathbf{j}}{c} \times \mathbf{B} \rightarrow \frac{\sigma}{c^2}\Omega r B^2 \qquad (16.5)$$

Think through the directions: this force will accelerate the flow, in the direction of motion of the rotating $\mathbf{B}$.

B. Channel flow with MHD: Hartmann flow

Here's a classic example of MHD effects on a fluid flow; think of a low-R_m, laboratory-type channel flow. We already saw pure-HD versions of this, in Chapter 9. Here, we revisit this situation with MHD effects added. We'll find that MHD effects can either drive the flow (a "pump", or drive a transverse current (a "generator"), depending on the cionfiguration of the experiment.

1. THE BASIC SETUP

The setup is: steady, viscous, incompressible, planar flow between plates, and across a $\mathbf{B}$ field. As in the case of Poiseuille flow, which we saw in Chapter 3, the flow here is maintained against friction by a pressure difference applied across the ends of the channel. The presence of a magnetic field will, however, change the nature of the solution.

The basic equations are

$$\begin{aligned} \nabla\times\mathbf{E} &= 0 \\ \nabla\cdot\mathbf{v} &= 0 \\ \eta\nabla\times\mathbf{B} &= \mathbf{E}+\mathbf{v}\times\mathbf{B} \\ \nabla p-\rho\nu\nabla^2\mathbf{v} &= \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} \end{aligned} \tag{16.6}$$

The first one is from Maxwell, with $\partial\mathbf{B}/\partial t=0$; the third is from the steady-state induction equation (think of "uncurling" $\partial\mathbf{B}/\partial t=0$. The other two should be familiar to you. Once again, ν is the kinematic viscosity, $\eta=c^2/4\pi\sigma$ is the diffusivity, and σ is the conductivity.

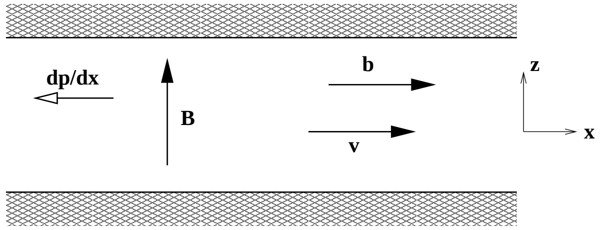

Figure 16.3. The geometry for Hartmann flow: a viscous conducting flow between parallel plates with a transverse B, and also a component $\mathbf{b}\parallel\mathbf{v}$. Following Somov figure 15.4

Figure 16.3 shows our setup. Let the plates lie at $z=\pm l$, with the flow along $\hat{\mathbf{x}}$. Things only vary with z (so the driving $\nabla p=dp/dx$) is constant along x, as before). The imposed field is $\mathbf{B}=(0,0,B_o)$. We look for induced fields $\mathbf{b}=(b(z),0,)$ and $\mathbf{v}=(v(z),0,0)$. In general, we allow a constant transverse electric field, $\mathbf{E}=E_o\hat{\mathbf{y}}$; but (you should be able to convince yourself that) if we hold the boundaries at a fixed potential, we will have $\mathbf{E}=0$ inside the flow.

2. THE SOLUTION

To proceed, ignore any $\mathbf{E}$ field (for reasons given above), and specify the system (16.6) to one dimension. We get

$$\begin{aligned} B_o v+\eta\frac{\partial b}{\partial z} &= 0 \\ \rho\nu\frac{\partial^2 v}{\partial z^2}+\frac{B_o}{4\pi}\frac{\partial b}{\partial z}-\frac{\partial p}{\partial x} &= 0 \\ \frac{\partial}{\partial z}\left(p+\frac{b^2}{8\pi}\right) &= 0 \end{aligned} \tag{16.7}$$

Play algebra on these, eliminate b, and find an equation for v:

$$\frac{dp}{dx}-\rho\nu\frac{d^2v}{dz^2}-\frac{B_o^2}{4\pi c\eta}v=0 \tag{16.8}$$

Because dp/dx is constant, we have a differential equation which can be solved for $v(z)$. If we apply no-slip boundary conditions, the solution can be written

$$v(z)=A\left(\cosh\mathrm{Ha}-\cosh\frac{\mathrm{Ha}z}{l}\right) \tag{16.9}$$

where A is an integration constant (which can be determined from the problem), and the *Hartmann number* has been defined as

$$\mathrm{Ha}=\frac{lB_o}{(4\pi\rho\nu\eta)^{1/2}} \tag{16.10}$$

How does this compare to the non-MHD channel flows in Chapter 2? Well, the velocity at the center of the flow is $v_o=v(z=0)=A(\cosh\mathrm{Ha}-1)$. The form (16.9) converts to the usual parabolic velocity profile, as $B\to 0$ (that is as $\mathrm{Ha}\to 0$:

$$v(z)\to v_o\left(1-\frac{z^2}{l^2}\right) \tag{16.11}$$

(Compare the channel-flow solutions in chapter 2). Alternatively, as Ha gets large, the solution flattens:

$$v(z)\to v_o\left[1-e^{-\mathrm{Ha}(l-z)/l}\right] \tag{16.12}$$

This is a much flatter velocity profile, with an exponential boundary layer close to the walls.

Note that everything else you might want for the solution – current, b, etc – can be found from the basic equations once you have the $v(z)$ solution.

3. AN MHD GENERATOR OR AN MHD PUMP?

What determines the on-axis flow, v_o? Combine (16.9) with Maxwell, to find the current density in the duct:

$$\frac{j_yB_o}{c}=\frac{\partial p}{\partial x}-\rho\nu A\left(\frac{\mathrm{Ha}}{l}\right)^2\cosh\frac{\mathrm{Ha}z}{l} \tag{16.13}$$

This integrates to give the total current, $I = \int j_y dz$:

$$\frac{IB_o}{c} = 2l\frac{\partial p}{\partial x} - 2\rho\nu A\left(\frac{\mathrm{Ha}}{l}\right)\sinh\frac{\mathrm{Ha}z}{l} \tag{16.14}$$

(Note, this latter is an implicit solution for the velocity amplitude A).

Thus, the Hartmann flow speed depends on the applied pressure gradient and the Lorentz force. This allows two regimes of operation. (1) If the flow is maintained by an external pressure gradient, the duct operates as an *MHD generator*, driving the transverse current I (this is useful if you somehow complete the circuit external to the duct..). (2) Alternatively, one can apply an external EM driver to create the transverse current I; interaction of this current with $\mathbf{B}_o$ gives rise to a Lorentz force that makes the plasma move along the duct, that is creates an *MHD pump*. This may be discussed further in class.

C. Magnetic Coupling of Two Plasma Slabs.

This is a quite different application, derived from space plasmas... think of the earth's ionosphere (which is partly ionized, partly neutral) and magnetosphere (which is fully ionized).

Picture: two plasma slabs, the top fully ionized, the bottom an ion-neutral mix, separated by some lower-density plasma. A vertical magnetic field threads both. Now let the top layer move at some $\mathbf{v}_o$; if it is close to ideal, flux freezing will pull the field lines along, and thus exert a transverse force on the lower layer. The system is governed, of course, by the momentum equation:

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \mathbf{j}\times\mathbf{B} - \rho\nu_{coll}\mathbf{v} \tag{16.15}$$

We allow a ∇p term, and have added a "friction" force, $\rho\nu_{coll}\mathbf{v}$; assuming some collisional coupling term ν_{coll}. In addition we anticipate a Lorentz force, $\mathbf{j}\times\mathbf{B}$.

We can analyze the system in two ways.

1. PURE MHD

The spirit here is, ignore currents and $\mathbf{E}$ fields; use Maxwell to convert $\mathbf{j}$ to $\mathbf{B}$. We thus write

$$\mathbf{j}\times\mathbf{B} \simeq -\nabla\frac{B_o^2}{8\pi} + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{16.16}$$

But now, we ignore gradients in the magnetic pressure (they are small), and estimate the second term as $(\mathbf{B}\cdot\nabla)\mathbf{B} \simeq (B_yB_o/H)\hat{\mathbf{y}}$. We thus have two force equations, for the top and bottom slabs:

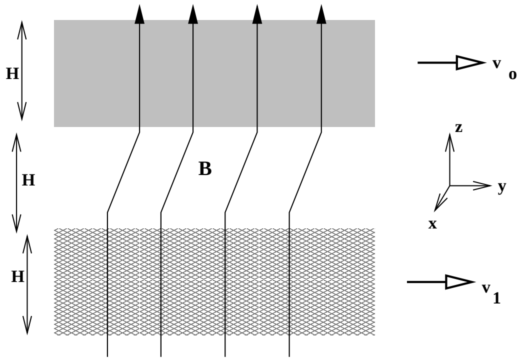

Figure 16.4. Two plasma slabs, in relative motion. The top slab moves at v_o, and is tied by $\mathbf{B}$ to the bottom slab. The slabs each have height H, which is also the inter-slab spacing. The slabs are infinite in the y direction, and have finite extent L_x in the x direction. Following Cravens Figure 8.19.

$$\begin{aligned}\rho_o\frac{Dv_o}{Dt} + \frac{dp}{dy} &= -\frac{B_yB_o}{4\pi H}\\ \rho_1\frac{Dv_1}{Dt} &= \frac{B_yB_o}{4\pi H} - \rho_1\nu_{coll}v_1\end{aligned} \tag{16.17}$$

The first equation describes the "drivers": either Dv_o/Dt or dp/dy must be finite, in order to drive the system. The second equation describes the response of the lower slab; clearly a steady state is possible there if $B_yB_o/4\pi H \simeq \rho_1\nu_{coll}v_1$ – in fact this will determine v_1, the velocity of the lower slab.

But: what determines B_y, i.e. the curvature (and tension) of the field lines? If we stick to the pure-MHD approach, we answer this using the induction equation (13.7). That is,

$$\begin{aligned}\nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B} &= 0\\ \frac{v_oB_o}{H} + \eta\frac{B_y}{H^2} &= 0\end{aligned} \tag{16.18}$$

where the second equation is again from scaling/dimensional analysis. We can thus estimate $B_y \simeq B_oHv_o/\eta$; and use this in (16.17) to solve the full system. NOTE that we must still pick the appropriate resistivity η for the bottom slab – Cravens argues for cross-field Pederson conductivity, using the same ion-neutral collision rate ν_{coll} that provides the frictional force on the lower slab.

2. CURRENT-BASED APPROACH

The above is a perfectly good solution...but Cravens notes that simple dimensional analysis may not be good enough when B_y is small (and presumably numerical methods may have trouble). We can also gain insight by repeating the analysis in terms of the currents and electric fields induced by the motion. We expect two cross-field currents, Pederson and Hall... The Pederson current is the interesting one.[1] The Lorentz force in the upper slab acts to decelerate it; that in the lower slab accelerates the slab:

$$\begin{aligned} \text{top}: \ \mathbf{j} \times \mathbf{B} &\simeq -j_{\perp o} B_o \hat{\mathbf{y}} \\ \text{bottom}: \ \mathbf{j} \times \mathbf{B} &\simeq j_{\perp 1} B_o \hat{\mathbf{y}} \end{aligned} \qquad (16.19)$$

The two momentum equations become

$$\begin{aligned} \rho_o \frac{Dv_o}{Dt} + \frac{dp}{dy} &= -j_{\perp o} B_o \\ \rho_1 \frac{Dv_1}{Dt} &= j_{\perp 1} B_o - \rho_1 \nu_{coll} v_1 \end{aligned} \qquad (16.20)$$

We can again find an equilibrium solution for the lower slab, which has $v_1 = j_{\perp 1} B_o / \rho_1 \nu_{coll}$.

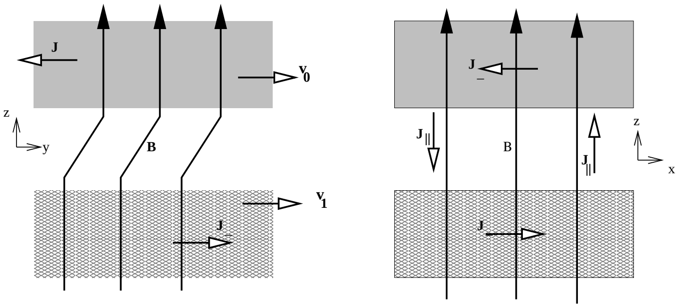

Figure 16.4. Two views of the problem, emphasizing the induced currents. Left, same view as in the first figure; induced $\mathbf{j} \times \mathbf{B}$ Lorentz forces as shown. The top slab carries $j_\perp$ out of the page, the bottom has $j_\perp$ into the page. The induced $\mathbf{E}$ is into the page throughout. Right, the same situation, but seen along the y axis. The complete induced current loop can be seen, with $j_\perp$ within the slabs (not labelled with subscripts due to limitations of Xfig), and field-aligned $j_{||}$ connecting the slabs. Following Cravens Figure 8.19.

Now: how do we find the current densities? First, consider current conservation, $\nabla \cdot \mathbf{j} = 0$ in a steady state. The Pederson current flows along $\hat{\mathbf{x}}$, and the slab is finite in that direction. Thus, the cross-field currents in the plasma slabs must be connected by field-aligned currents inbetween the slabs, as shown in the figure. In addition, currrent conservation requires $j_{\perp o} H = j_{\perp 1} H$, or $j_{\perp 0} = j_{\perp 1}$ (because we've chosen the two slabs to have the same thickness). We can therefore combine the results above, as

$$v_1 = -\frac{1}{\rho_1 \nu_{coll}} \left(\rho_o \frac{Dv_o}{Dt} + \frac{dp}{dy} \right) \qquad (16.21)$$

which allows, again, the possibility of the two drivers, Dv_o/Dt and dp/dy.

To finish this, we still need to find $j_{\perp o}$. Consider the top slab: if it is collisionless, we expect $\mathbf{E} = -\mathbf{v} \times \mathbf{B}_o = -v_o B_o \hat{\mathbf{x}}$ in the top slab. But also, because this is a steady-state, one-dimensional problem, we have $\nabla \times \mathbf{E} = 0$, thus $\mathbf{E}$ must be the same in the bottom slab (as well as in the intervening space). It follows that the $\mathbf{B}_o$ lines "map" the field from one slab to the other: the magnetic field lines are equipotentials. We can, therefore, use Ohm's law (for the Pederson current) to find $j_{\perp o} = j_{\perp 1} = \sigma_\perp v_o B_o$..thus finishing the problem.

3. ENERGETICS AND EQUIVALENT CIRCUIT

Finally, consider the energetics of this example. We can think of $\mathbf{j} \cdot \mathbf{E}$ as a "local" source of EM energy. In this example, $\mathbf{j}_o \cdot \mathbf{E} > 0$ while $\mathbf{j}_1 \cdot \mathbf{E} < 0$. Thus, we can think of the top slab as an MHD generator, which produces energy that is dissipated in the lower slab, the resistor. The power per unit length generated (top) or dissipated (lower) is $\int \mathbf{j} \cdot \mathbf{E} dx dz \simeq L_x H v_o B_o j_{\perp o}$. This is also, nicely, the same as the power transferred from top to bottom by the Poynting flux: $\int (\mathbf{E} \times \mathbf{B}) 4\pi^{-1} dx dy \simeq v_o B_p B_y L_x / 4\pi$ (check: you can verify this is the same result). We can also describe this problem in terms of the total circult and a global resistance...let $\mathcal{E} \simeq v_o B_o L_x$ be the EMF generated (volts); the resistance of the lower slab is $R = L_x / \sigma_\perp H L_y$ (ohms); and the current $I = /R$ (amps), as it should be.

References

I've taken the magnetic damping/stirring material mostly from Davidson. My Hartmann flow discussion follows Woods & Somov. The plasma-coupling discussion follows Cravens.

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[1] the Hall current is there, in principle infinite in extent, so without gradients, and not interesting for this problem.