17. WAVES AND SHOCKS IN MHD

Nonmagnetized flows have one important characteristic signal speed – that's the *sound speed*, as we saw in Chapter 7. When we add a $\mathbf{B}$ field, things get more interesting ... there is more than one possible "signal-carrying" wave.

A. MHD Waves

In Chapter 7 we found that a density perturbation propagates at a speed $c_s = (\partial p/\partial \rho)^{1/2}$. This is a critical quantity in fluid dynamics – the speed at which information can propagate in a gas. For a magnetized plasma, the situation is (not surprisingly) more complex. There are two characteristic waves, Alfven and magnetosonic. To start, we carry out a linear analysis to discover what types of perturbations propagate, and at what speeds.

1. BASIC STRUCTURE: LINEAR ANALYSIS

Start with a set of basic, unperturbed equations:

$$\begin{gathered}
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \\
\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \\
\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \\
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \\
\nabla \cdot \mathbf{B} = 0
\end{gathered} \tag{17.1}$$

Linearize these in the usual way. That is, take

$$\begin{gathered}
\rho = \rho_o + \rho_1 \; ; \quad \mathbf{v} = \mathbf{v}_1 \\
p = p_o + p_1 \; ; \quad \mathbf{B} = \mathbf{B}_o + \mathbf{B}_1
\end{gathered} \tag{17.2}$$

where the unperturbed ("zero" subscript) state is taken as uniform and homogeneous. (The more general case of background structure, won't be addressed here). With this, we get

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\rho_o + \rho_o(\nabla \cdot \mathbf{v}_1) = 0 \\
&\rho_o \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_o \\
&\frac{\partial p_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)p_o - c_s^2\left(\frac{\partial \rho_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\rho_o\right) = 0 \\
&\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_o) \\
&\nabla \cdot \mathbf{B}_1 = 0
\end{aligned} \tag{17.3}$$

where $c_s^2 = \gamma k_B T_o/m$ is the square of the (unperturbed) sound speed.

These equations (17.3) can be combined, algebraed, and eventually made into one equation for the disturbance velocity, $\mathbf{v}_1$:

$$\begin{aligned}
\frac{\partial^2 \mathbf{v}_1}{\partial t^2} &= c_s^2 \nabla(\nabla \cdot \mathbf{v}_1) \\
&= [\nabla \times (\nabla \times (\mathbf{v}_1 \times \mathbf{B}_o))] \times \frac{\mathbf{B}_o}{4\pi\rho_o}
\end{aligned} \tag{17.4}$$

If we now assume a plane-wave solution,

$$\mathbf{v}_1(\mathbf{r}, t) = \mathbf{v}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{17.5}$$

our linearized equation reduces to

$$\begin{aligned}
\omega^2 \mathbf{v}_1 =& c_s^2 \mathbf{k}(\mathbf{k} \cdot \mathbf{v}_1) \\
&+ [\mathbf{k} \times (\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_o))] \times \frac{\mathbf{B}_o}{4\pi\rho_o}
\end{aligned} \tag{17.6}$$

Now, finally, we're in position to analyze basic wave modes for this system.

As a prelude, note that if $\mathbf{B}_o = 0$, (17.6 reduces to

$$\omega^2 \mathbf{v}_1 = c_s^2 \mathbf{k}(\mathbf{k} \cdot \mathbf{v}_1) \tag{17.7}$$

Dotting this with $\mathbf{k}$, and assuming $\mathbf{k} \cdot \mathbf{v}_1 \neq 0$ (*i.e.*the perturbed velocity has some component parallel to the wavevector), this gives

$$\omega = c_s k \tag{17.8}$$

for the dispersion relation. This should make you feel good – we have recovered sound waves.

2. ALFVEN WAVES

These are waves in which the magnetic field dominates. It exerts the restoring force; fluctuations in the plasma density and pressure are either exactly zero, or unimportant. We can start by guesstimating the likely wave speed. Recall that waves in an elastic wire propagate due to the tension; as the field lines in a plasma exert a tension $B_o^2/4\pi$, one might expect a wave speed

$$v_A = \frac{B_o}{(4\pi\rho_o)^{1/2}} \tag{17.9}$$

This is the *Alfven speed*, and it is, indeed, a useful scaling speed for waves in a magnetized plasma. We can also note directly that $\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{B}_1 = 0$; so that the magnetic field perturbation must be normal to the wavevector.

Following Priest, we now specify to the magnetized-only limit, that is, dropping the p_o terms from (17.3). This is called the cold-plasma limit, and means we're ignoring any internal-pressure effects (wo we no longer have sound waves). Doing this, rewriting to make the angle terms explicit letting θ be the angle between $\mathbf{k}$ and $\mathbf{B}_o$), and letting $\hat{\mathbf{b}}_o$ be the unit vector along $\mathbf{B}_o$, we have the Alfvenic wave dispersion relation:

$$\frac{\omega^2}{v_A^2}\mathbf{v}_1 = k^2\cos^2\theta\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)k\cos\theta\hat{\mathbf{b}}_o + \left[(\mathbf{k}\cdot\mathbf{v}_1) - k\cos\theta(\hat{\mathbf{b}}_o\cdot\mathbf{v}_1)\right]\mathbf{k} \qquad (17.10)$$

Dotting this with $\hat{\mathbf{b}_o}$ gives $\hat{\mathbf{b}_o}\cdot\mathbf{v}_1 = 0$; thus, waves in this limit just have the perturbed velocity normal to the ambient magnetic field. Dotting this with $\mathbf{k}$ gives

$$\left(\omega^2 - k^2v_A^2\right)\mathbf{k}\cdot\mathbf{v}_1 = 0 \qquad (17.11)$$

This has two separate solutions, as follows.

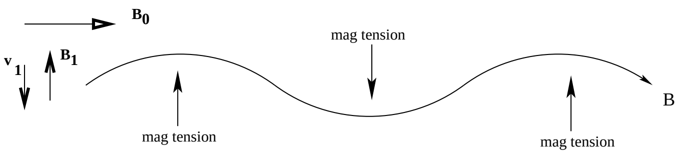

Figure 17.1. Schematic of (shear) Alfven waves; the perturbed B_1 and v_1 terms are perpendicular to the background field B_o. Following Cravens Figure 4.16.

• **(shear) alfven waves**. This is the case of an incompressible perturbation:

$$\nabla\cdot\mathbf{v}_1 = 0\,; \qquad \mathbf{k}\cdot\mathbf{v}_1 = 0 \qquad (17.12)$$

which is, of course, one solution to (17.11). These are, thus, transverse waves (particle motion transverse to $\mathbf{k}$, $\mathbf{B}_1 \perp \mathbf{B}_o$). Using this, (17.10) gives

$$\omega = kv_A\cos\theta \qquad (17.13)$$

for *Alfven waves* (sometimes called shear Alfven waves). These waves have a phase speed $v_A\cos\theta$, which agrees with our simple argument for waves propagating exactly along $\mathbf{B}_o$. Putting this solution back into the linearized equations (17.3), the first two show that there are no density or pressure perturbations: $\rho_1 = p_1 = 0$. So, the plasma just moves back and forth with $\mathbf{B}_1$, without any compressive effects. (The same equation sshow that $\mathbf{v}_1 = \mathbf{B}_1/\left(4\pi\rho_o\right)^{1/2}$, so that $\mathbf{v}_1$ and $\mathbf{B}_1$ are in the same direction).

An interesting aside is that these solutions also describe finite-amplitude waves. Most wave solutions require small-amplitude perturbations (as we began with here). However, with some algebra one can verify that an Alfvenic disturbance described by

$$\mathbf{v}_1 = -\frac{\mathbf{B}_1}{\left(4\pi\rho_o\right)^{1/2}}\,; \qquad |\mathbf{B}_o + \mathbf{B}_1| = \text{constant} \qquad (17.14)$$

satisfies the full equations, 17.1), without the need to linearize. One important consequence of these waves, Priest notes, is that finite-amplitude waves do not tend to steepen, and so dissipate must less readily than other wave modes.

• **compressional alfven waves** The second solution to (17.11) is

$$\omega = kv_A \qquad (17.15)$$

which describes *compressional Alfven waves*. For these waves, the linearized equations (17.3) show that $\mathbf{v}_1$ is normal to $\mathbf{B}_o$, and lies in the $(\mathbf{k}, \mathbf{B}_o)$ plane. It therefore has components both along and transverse to $\mathbf{k}$, and so gives rise to density and pressure fluctuations.

3. MAGNETOSONIC WAVES

When the gas pressure is dynamically comparable to the magnetic field, the wave nature is different. We might expect the wave speed to be a mixture of compressive effects (through c_s) and magnetic effects (through v_A).

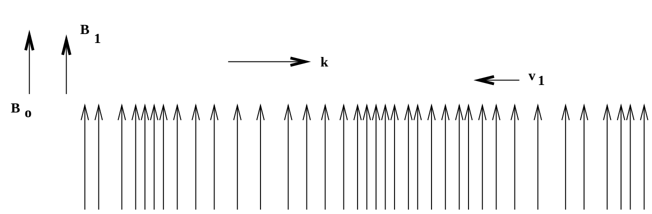

Figure 17.2. Schematic of magnetosonic waves, illustrating a compressive wave propagating at right angles to the background B_o. Following Cravens figure 4.17.

Returning to (17.6) and keeping the pressure and compresssive terms, the ruling equation here is

$$\frac{\omega^2}{v_A^2}\mathbf{v}_1 = k^2\cos^2\theta\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)k\cos\theta\hat{\mathbf{b}_o} + \left[(1+c_s^2/v_A^2)(\mathbf{k}\cdot\mathbf{v}_1) - k\cos\theta(\hat{\mathbf{b}_o}\cdot\mathbf{v}_1)\right]\mathbf{k} \tag{17.16}$$

Dotting this with $\mathbf{k}$ and $\mathbf{b}_o$, in turn, gives two useful relations:

$$\left(-\omega^2 + k^2c_s^2 + k^2v_A^2\right)(\mathbf{k}\cdot\mathbf{v}_1) = k^3v_A^2\cos\theta(\hat{\mathbf{b}_o}\cdot\mathbf{v}_1) \tag{17.17}$$

and

$$k\cos\theta c_s^2(\mathbf{k}\cdot\mathbf{v}_1) = \omega^2(\hat{\mathbf{b}_o}\cdot\mathbf{v}_1) \tag{17.18}$$

If $\mathbf{k}\cdot\mathbf{v}_1 = 0$, we recover the Alfven wave solution (17.13). If this isn't zero, we can combine these two relations to find the dispersion relation for *magnetosonic waves*:

$$\omega^4 - \omega^2k^2\left(c_s^2 + v_A^2\right) + c_s^2v_A^2k^4\cos^2\theta = 0 \tag{17.19}$$

For forward-pointing ($\omega/k > 0$) waves, there are two distinct solutions:

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left(c_s^2 + v_A^2\right) \pm \frac{1}{2}\left(c_s^4 + v_A^4 - 2c_s^2v_A^2\cos 2\theta\right)^{1/2} \tag{17.20}$$

The minus sign in (17.20) gives what is known as the *slow mode*; the plus sign gives the *fast mode*. The Alfven speed lies inbetween these two wave speeds; thus the Alfven wave is occasionally called the *intermediate mode*.

These two magnetosonic modes may be thought of (as Priest notes) as a sound wave, modified by the magnetic field, and a compressional Alfven wave, modified by the plasma pressure. If the B field becomes small ($v_A \to 0$), the slow mode disappears, and the fast wave becomes a sound wave. If the gas pressure becomes small ($c_s \to 0$), the slow mode disappears again, and the fast wave becomes a compressional Alfven wave.

4. VALIDITY OF MHD WAVE THEORY

An important limitation on this MHD wave theory is that it assumes simple, collective particle motion. At higher frequencies, the single-particle motions in mixed magnetic and electric fields cannot keep up with the driving wave. The wave modes become more complicated as a result, and require a plasma-physical approach (following individual particles, or at least individual species). The common usage is that we can treat waves by MHD methods for wave frequencies below $\Omega_i = eB_o/m_ic$, the ion gyrofrequency.

B. MHD Shocks; Jump Conditions

In Chapter 9, we found that density perturbations – sound waves – will steepen and can develop into shocks. Just as there are more than one signal-carrying type of MHD wave, there are more than one type of MHD shock.

Also in chapter 9, we applied conservation laws to determine the jump conditions at hydrodynamic shocks. We will follow the same path here, and we will find that there is more than one soltuion to the jump conditions – more than one type of shock. In particular, magnetosonic waves are compressive, and can steepen into shocks; the two MS modes, fast and slow, connect to two types of MHD shocks. Alfven waves, not being compressive, do not steepen into shocks (although there are formal discontinuities that one can find, associated with Alfven waves.)

We work in a frame in which the shock is at rest; this makes everything steady state. We also ignore dissipation; it matters *within* the shock, but here we idealize to an infintely thin jump. We use to the notation of Chapter 9, that is $[\![A]\!]$ is the jump in A across a boundary. The unit vectors are $\hat{\mathbf{n}}$, normal to the boundary, and $\hat{\mathbf{t}}$, tangential to it. We start with the basic equations, and for each write down the jump across the shock.

• **Maxwell**. We know $\nabla\cdot\mathbf{B} = 0$; this

$$[\![\mathbf{B}\cdot\hat{\mathbf{n}}]\!] = 0\,; \quad B_n = \text{constant} \tag{17.21}$$

is the continuity of the normal component, B_n, of the $\mathbf{B}$ field. (You recall that the jump in B_t can be finite if there is a surface current).

• **Mass flux**. We have, in a steady state, $\nabla\cdot(\rho\mathbf{v}) = 0$, so that

$$[\![\rho\mathbf{v}\cdot\hat{\mathbf{n}}]\!] = 0\,; \quad \rho v_n = \text{constant} \tag{17.22}$$

is the continuity of mass flux.

• **Induction**. In steady state, $\nabla\times(\mathbf{v}\times\mathbf{B}) = 0$. Across the boundary, this gives

$$[\![(\mathbf{v}\times\mathbf{B})_t]\!] = 0\,; \quad [\![v_n\mathbf{B}_t]\!] = B_n\,[\![\mathbf{v}_t]\!] \tag{17.23}$$

where we've used B_n =constant in the last.

• **Momentum flux**.

$$\left[\!\!\left[\rho\mathbf{v}(\mathbf{v}\cdot\hat{\mathbf{n}}) + p\hat{\mathbf{n}} - \left(\mathbf{B}\frac{\mathbf{B}\cdot\hat{\mathbf{n}}}{4\pi} - \frac{B^2}{4\pi}\hat{\mathbf{n}}\right)\right]\!\!\right] = 0 \tag{17.24}$$

is the basic momentum flux. A useful alternative to this (resembling Bernoulli's equation) comes from dotting it with $\hat{\mathbf{n}}$:

$$\left[\!\!\left[p + \rho(\mathbf{v}\cdot\hat{\mathbf{n}})^2 + \frac{1}{8\pi}\left((\mathbf{B}\cdot\hat{\mathbf{t}})^2 - (\mathbf{B}\cdot\hat{\mathbf{n}})\right)^2\right]\!\!\right] = 0 \tag{17.25}$$

We can also scalar multiply with $\hat{\mathbf{t}}$ to get

$$\left[\!\!\left[\rho v_n \mathbf{v}_t - \frac{B_n\mathbf{B}_t}{4\pi}\right]\!\!\right] = 0 \tag{17.26}$$

These last two can be turned into more useful forms if we isolate the (conserved) mass flux, ρv_n:

$$[\![p]\!] + \rho^2 v_n^2\left[\!\!\left[\frac{1}{\rho}\right]\!\!\right] + \frac{1}{8\pi}\left[\!\!\left[B_t^2\right]\!\!\right] = 0 \tag{17.27}$$

and

$$\rho v_n [\![\mathbf{v}_t]\!] - \frac{B_n}{4\pi}[\![\mathbf{B}_t]\!] = 0 \tag{17.28}$$

• **Energy flux**. There are several useful forms here. A basic form is

$$\left[\!\!\left[\frac{1}{2}\rho v^2\mathbf{v} + \frac{\gamma}{\gamma-1}p\mathbf{v} - \frac{1}{4\pi}(\mathbf{v}\times\mathbf{B})\times\mathbf{B}\right]\!\!\right] = 0 \tag{17.29}$$

As before (chapter 6), we can define $h = e + p/\rho = \gamma p/(\gamma-1)\rho$, with $e = p/(\gamma-1)\rho$; and again use continuity of B_n and mass flux, to write this as

$$\rho v_n\left[\!\!\left[\frac{v_n^2}{2} + h + \frac{v_t^2}{2} + \frac{B_t^2}{4\pi\rho}\right]\!\!\right] = \frac{B_n}{4\pi}[\![\mathbf{v}_t\cdot\mathbf{B}_t]\!] \tag{17.30}$$

As before, the nature of the shock depends on its orientation relative to the flow; and also, in this case, on its orientation to the $\mathbf{B}$ field. We can think of three cases.

• *Parallel Shocks* have the shock normal parallel to $\mathbf{B}$. These are no different from unmagnetized shocks. The fluid can flow along the $\mathbf{B}$ field without hindrance. There is no $\mathbf{v}\times\mathbf{B}$ term to maintain an EMF, nor is there a current (as $\mathbf{B}$ is constant by assumption).
• *Perpendicular (Normal) Shocks* have the shock normal perpendicular to $\mathbf{B}$. These are the simplest cases of magnetized shocks.
• *Oblique Shocks* are, as before, the most complicated. My presentation follows Priest, and Woods; the latter is the most thorough. There does not seem to be an extensive literature on oblique MHD shocks – the complexity of the results may be why.

C. Perpendicular (Normal) Shocks

That is, let $B_n = 0$, but allow non-zero v_n. A useful fact: from (17.28), we know $[\![\mathbf{v}_t]\!] = 0$ in this case. We can then, always, transform to a frame with $\mathbf{v}_t = 0$; this simplifies things (and I drop the subscript on v_n). The jump conditions now become, for perpendicular shocks,

$$\begin{aligned}\rho_1 v_1 &= \text{constant}\\ B_1 v_1 &= \text{constant}\\ \rho_1 v_1^2 + p_1 + \frac{B_1^2}{8\pi} &= \text{constant}\\ \frac{1}{2}v_1^2 + h_1 + \frac{B_1^2}{4\pi\rho} &= \text{constant}\end{aligned} \tag{17.31}$$

where "constant" means the quantity is the same upstream (subscript 1) and downstream (subscript 2) of the shock.

The geometry is illustrated in the Figure.

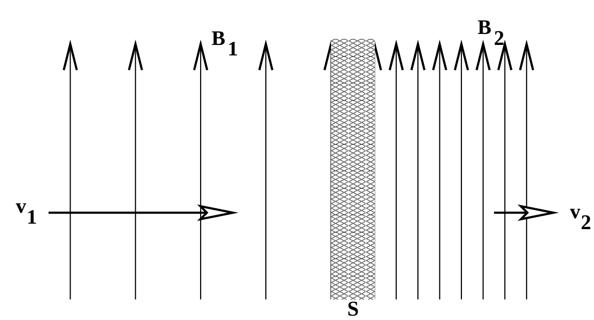

Figure 17.3. A plane perpendicular shock; the magnetic field is parallel to the shock front, and perpendicular to the shock normal.

Again, let's call the compression ratio $X = \rho_2/\rho_1$ (where "2" is downstream and "1" is upstream"), and let the upstream B field be scaled by $b = B^2/4\pi\rho c_s^2 = v_A^2/c_s^2$. The various equations can be written as,

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= X \\
\frac{v_1}{v_2} &= X \\
\frac{B_2}{B_1} &= X \\
\frac{p_2}{p_1} &= \gamma\mathcal{M}^2\left(\frac{X-1}{X}\right) + 1 + \frac{\gamma b}{2}\left(1 - X^2\right)
\end{aligned} \tag{17.32}$$

where the compression ratio is the solution of

$$\begin{aligned}
&b\left[(\gamma-2)X^2 - \gamma X\right] \\
&\quad = X\left[(\gamma-1)\mathcal{M}^2 + 2\right] - (\gamma+1)\mathcal{M}^2
\end{aligned} \tag{17.33}$$

Solutions to this are shown in Figure 17.4. (Details: doing the algebra on equations (17.32), and eliminating p_2/p_1, gives a cubic equation in X. I've factored out $(X-1)$ (which is not zero in any interesting case), to get this quadratic. Note also typos in two of Priest's equations have been fixed.)

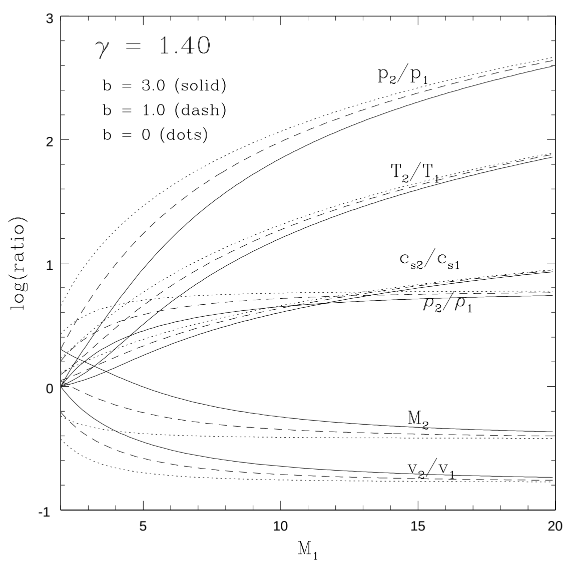

Figure 17.4. Numerical solution of the jump conditions, 17.11, for unmagnetized shocks and for two different values of $b = v_A^2/c_s^2$. See text for discussion.

Solving (17.33) shows us that a perpendicular, magnetized shock is compressive only if the upstream flow (relative to the shock) is above the fast magnetosonic speed. That is, $X \geq 1$, if $\mathcal{M}^2 \geq 1 + v_A^2/c_s^2$; the upstream flow must satisfy $v_1^2 \geq c_{s1}^2 + v_{A1}^2$. Compared to the HD solutions (Chapter 7), the effect of the B field is to reduce the density jump X below its hydrodynamic value (since the flow energy can go to magnetic as well as thermal energy). At the same time, the pressure and temperature jumps are larger than for the $B = 0$ case. These shocks have the same limiting behavior as $\mathcal{M} \to \infty$: the density and field jumps asymptote to $(\gamma+1)/(\gamma-1)$; while the pressure and temperature jumps rise without bound.

D. Oblique Shocks

MHD shocks at general angles can be more complex, of course. To specify the geometry, let $\hat{\mathbf{x}}$ be the direction along the shock normal, and $\hat{\mathbf{y}}$ be in the plane of the shock, and in the plane of the $\mathbf{B}$ field.

1. DO IT GENERALLY

The general jump conditions (or conservation laws) are as follows. For mass

$$\rho v_x = \text{constant} \tag{17.34}$$

(where "constant" means has the same value on both sides of the shock). The $\nabla \cdot \mathbf{B}$ Maxwell-eqn becomes

$$B_x = \text{constant} \tag{17.35}$$

Two components of the force equation become

$$p + \frac{B^2}{8\pi} - \frac{B_x^2}{4\pi} + \rho v_x^2 = \text{constant} \tag{17.36}$$

and

$$\rho v_x v_y - \frac{B_x B_y}{4\pi} = \text{constant} \tag{17.37}$$

The induction equation becomes

$$v_x B_y - v_y B_x = \text{constant} \tag{17.38}$$

And, finally, the energy equation becomes

$$v_x\left(p + \frac{B^2}{8\pi}\right) - B_x\frac{\mathbf{B}\cdot\mathbf{v}}{4\pi} + v_x\left(\rho e + \frac{1}{2}\rho v^2 + \frac{B^2}{8\pi}\right) = \text{constant} \tag{17.39}$$

These equations can, in principle, be solved for the general shock jump conditions. But they are long and complex, so most authors simplify by working in a useful reference frame.

2. FIND A USEFUL REFERENCE FRAME

At this point, it is very helpful to transform to a frame moving along the shock face, at a speed

$$v_{1y} = v_{1x} \frac{B_{1y}}{B_{1x}} \tag{17.40}$$

Such a transform can always be found, if $B_x \neq 0$. In this frame, the plasma velocity is parallel to the B field on both sides of the shock; which in turn implies that $\mathbf{E} = 0$ on both sides.

This transform simplifies the situation, but the algebra is still complex. It also turns out that three solutions are possible: *fast shocks* (which develop from fast-mode MS waves); *intermediate shocks* (which develop from Alfven waves); and *slow shocks* (which develop from slow-mode MS waves). I've seen two different analytic approaches, & will summarize both here.

3. NOW SOLVE THE SYSTEM: I

To follow convention, and shorten the notation a bit, we define $\nu = 1/\rho$ as the inverse density ("specific volume"). Two important constants, across the shock, are $m = \rho v = v/\nu$, the normal mass flux; and B_x, the normal field component. Recall $[\![A]\!] = A_2 - A_1$ is the jump in A; and $\langle A \rangle = \frac{1}{2}(A_1 + A_2)$ is the mean value of A. The jump conditions are, now,

$$[\![B_x]\!] = 0 \tag{17.41}$$

$$[\![v_x]\!] - m\,[\![\nu]\!] = 0 \tag{17.42}$$

$$n\,[\![v_x]\!] + [\![p]\!] + \frac{1}{4\pi}\langle B_y \rangle\,[\![B_y]\!] = 0 \tag{17.43}$$

$$n\,[\![v_y]\!] - \frac{1}{4\pi} B_x\,[\![B_y]\!] = 0 \tag{17.44}$$

$$\frac{1}{2}\left[\!\left[v_x^2 + v_y^2\right]\!\right] + [\![h]\!] = 0 \tag{17.45}$$

and

$$B_x\,[\![b_y]\!] - m\,[\![\nu]\!]\,\langle B_y \rangle - m\langle \nu \rangle\,[\![B_y]\!] = 0) \tag{17.46}$$

To solve this system, Woods defines θ_1 as the angle between $\mathbf{B}_1$ and the shock normal; andintroduces the "useful intermediate variables"

$$\begin{aligned} \mathcal{S} &= \frac{c_s^2}{v_A^2}\,; \qquad P = \frac{[\![p]\!]}{p_1} \\ \epsilon &= \frac{[\![B_y]\!]}{B_1}\,; \qquad \chi = \frac{1}{\gamma}\frac{\mathcal{S}}{\epsilon}P + \frac{1}{2}\epsilon \end{aligned} \tag{17.47}$$

In these variables, the jump conditions combine to a quadratic (taking $\gamma = 5/3$ explicitly):

$$\left(2\sin\theta_1 - \frac{2}{3}\epsilon\right)\chi^2 - 2\left(\frac{5}{6}\epsilon\sin\theta_1 - 1 + \mathcal{S}_1\right)\chi - (\epsilon + 2\mathcal{S}_1\sin\theta_1) - 0 \tag{17.48}$$

• **method # 1**. which can be solved directly for χ, given choices of $\mathcal{S}_1$ and θ_1 (that is, choices for the upstream magnetic field ratio and the flow angle). Slow shocks correspond to $\chi < 0$, and fast shocks to $\chi > 0$. The more usual ratios can be found from the χ solution:

$$\frac{\rho_2}{\rho_1} = 1 + \epsilon\frac{\chi + \sin\theta_1}{1 + \chi\sin\theta_1} \tag{17.49}$$

$$\frac{p_2}{p_1} = 1 + \frac{5\epsilon}{3\mathcal{S}_1}\left(\chi - \frac{1}{2}\epsilon\right) \tag{17.50}$$

and

$$\frac{v_{x1}^2}{b_{x1}^2} = 1 + \frac{\sin\theta_1}{\cos^2\theta_1}(\chi + \sin\theta_1) + \frac{\epsilon}{\cos^2\theta_1}(\chi + \sin\theta_1) \tag{17.51}$$

Woods presents a long discussion of the details of this system and some solutions.

4. NOW SOLVE THE SYSTEM: II

Alternatively, the conservation laws, (17.41)-(17.46), can be written as

$$\begin{aligned} \rho v_x &= \text{constant} \\ B_x &= \text{constant} \\ \rho v \mathbf{v} = \left(p + \frac{B_y^2}{8\pi}\right)\hat{\mathbf{n}} - \frac{B_x}{4\pi}\mathbf{B} &= \text{constant} \\ \frac{1}{2}(v_x^2 + v_y^2) + h &= \text{constant} \end{aligned} \tag{17.52}$$

(recalling that $h = \gamma p/(\gamma - 1)\rho$ is the enthalpy).

To solve these, we can (following Priest) again use the compression ratio, $X = \rho_2/\rho_1$. Keeping our definitions, $c_s^2 = \gamma p/\rho$, and $v_A^2 = B^2/4\pi\rho$ (all evaluated upstream), the jump conditions become

$$\begin{aligned} \frac{v_{2x}}{v_{1x}} &= \frac{1}{X} \\ \frac{v_{2y}}{v_{1y}} &= \frac{v_1^2 - v_{A1}^2}{v_1^2 - X v_{A1}^2} \\ \frac{\rho_2}{\rho_1} &= X \\ \frac{B_{2y}}{B_{1y}} &= \frac{v_1^2 - v_{A1}^2}{v_1^2 - X v_{A1}^2} X \\ \frac{p_2}{p_1} &= X + \frac{(\gamma - 1) X v_1^2}{2 c_{s1}^2}\left(\frac{X^2 - 1}{X^2}\right) \end{aligned} \tag{17.53}$$

where θ is the angle between the upstream magnetic field and the shock normal. The compression ratio X is now the solution of

$$\begin{aligned} &\left(v_1^2 - X V_{A1}^2\right) F_1(X, \theta) \\ &\quad + \frac{1}{2} v_{A1}^2 v_1^2 \sin^2\theta X F_2(X) = 0 \end{aligned} \tag{17.54}$$

where

$$\begin{aligned} F_1(X, \theta) &= X c_{s1}^2 \\ &\quad + \frac{1}{2} v_1^2 \cos\theta \left[X(\gamma - 1) - (\gamma + 1)\right] \end{aligned} \tag{17.55}$$

and

$$\begin{aligned} F_2(X) &= \left(\gamma + X(2 - \gamma)\right) v_1^2 \\ &\quad - X v_{A1}^2 \left((\gamma + 1) - X(\gamma - 1)\right) \end{aligned} \tag{17.56}$$

Presumably this equation could be solved numerically (picking values of $\cos\theta$ of $\mathcal{M} = v_1/c_{s1}$ and also of v_{A1}/c_{s1}). However, I have not seen such solutions presented, nor have I felt the need to work them out myself.

5. BACK TO THE PHYSICS

Whichever solution method one uses, three solutions of these equations can be found.

One has $X = 1$, is called an *intermediate* or *Alfven* wave, and isn't of great interst as it isn't compressive.[1] In the intermediate wave, the tangential field component is simply reversed in sign, so the overall field rotates without changing magnitude; hence the name.

The other two solutions are compressive, with $X > 1$, and are *slow* and *fast* shocks. The *slow shock* has $v_1^2 \leq v_{A1}^2$, and also has $B_2 < B_1$. That is, the magnetic field bends *towards* the shock normal in a slow shock. Similarly, in the slow shock the parallel flow is slowed down, $v_{2y} < v_{1y}$. Conversely, the *fast shock* has $v_1^2 \geq X V_{A1}^2$, and so $B_2 > B_1$. Thus, the field bends *away from* the shock normal in a fast shock. The parallel flow is speeded up in the fast shock, $v_{2y} > v_{1y}$.

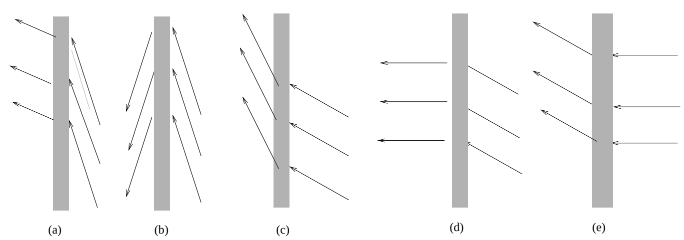

Figure 17.5. Left, the changes in magnetic field direction that are caused by the three types of oblique MHD shock waves; (a), slow mode shock, in which $\mathbf{B}$ bends toward the shock normal; (b) intermediate wave, in which $\mathbf{B}$ "flips"; and (c) fast shock, in which $\mathbf{B}$ bends away from the shock normal. From Priest figure 5.6. Right, the two special cases of switch-off (d) and switch-on (e) shocks. From Priest figure 5.7.)

There are two particular special cases of these. The slow solution becomes a *switch-off shock* when $v_1 = v_{A1}$. In this limit, the tangential field vanishes behind the shock, $B_{2y} = 0$. The fast solution becomes a *switch-on shock* when $v_1 = X^{1/2} v_{A1}$. In this limit, the upstream field is parallel to the shock normal: $B_{1y} = 0$.

References

The two best references I know for MHD waves and shocks are Priest and Woods. I've followed both of them here.

[1] Apparently it was the first one found, and was at that point of interest for laboratory plasmas; so I gather from Bazer & Ericson.