18. MAGNETIC RECONNECTION

Magnetic reconnection is an innocent-sounding name for an important, and initially unintuitive, process. It can be defined as the breaking, and topological rearrangement, of $\mathbf{B}$ field lines. That means, it violates what we've learned about magnetic flux conservation – which says that the $\mathbf{B}$ lines are "frozen into" the plasma. Magnetic flux is conserved only when resistive dissipation can be ignored; reconnection is an effect of localized, intense dissipation regions. It results in magnetic energy being converted to plasma energy (both heat and bulk motions).

Because reconnection allows field lines to be rearranged, and to "move through" a plasma, it can "connect" regions of plasma which had previously been magnetically isolated. To be specific: think of space plasma, where the electrical conductivity is very high (all those free charges can move where they want to), and therefore the flux freezing limit is a good one. This means that the plasma particles can freely move and mix along field lines, but cannot do so across the field. A particle always remains tied to the same field line as it moves with the plasma flow. Think, then, about two initially separate plasma regions which come into contact with one another.[1] It follows that the two plasmas will not mix; rather, a thin *boundary layer* will form between them, which keeps them quite separate. (The location of the boundary layer is, of course, determined by pressure balance and large-scale fluid dynamics.) In general, the magnetic fields on either side of the boundary layer will have different strengths and orientations, so that a *current sheet* must exist within the boundary layer.

Thus, flux freezing leads inevitably to the prediction that in plasma systems space will be divided into separate cells, which contain magnetized plasma from different sources, separated from each other by thin current sheets. Based on what we've seen up to now, the plasmas from these separate domains should be able to mix only very slowly, as resistivity dissipates the current and magnetic flux. Looking ahed to (18.5), this should happen on the (usually very slow) timescale, $\tau_{diss} \sim L^2/\eta$. But: the answer is much more interesting. A mixture of dynamics and resistivity allows the topology of the magnetic field to change, and the plasmas to mix, on a much shorter timescale. This process is called *magnetic reconnection*, and quite a culture has evolved around it. In this chapter I'll try to give an overview of this culture, both the simple treatments often found in the literature, and an impression of what might realy happen.

[1] To be specific, think of the solar wind where it encounters the earth's magnetosphere. Both of these are diffuse, magnetized, flux-frozen plasmas.

Space examples are often quoted – the interaction of the solar wind with the earth's magnetosphere, or the rapid energy release that creates a solar flare. But reconnection is also important in laboratory plasmas, where it underlies the "Taylor relaxation" process we talked about in Chapter 15; and it is critical to the existence of the dynamos that are responsible for the earth's and sun's magnetic fields.

A. Advection vs. dissipation of magnetic field

To set the stage, think back to the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \qquad (18.1)$$

The first term on the right-hand side describes inductive effects of the plasma flow on the B field; the second term describes resistive diffusion or dissipation. The ratio of the two terms is given by the magnetic Reynolds number,

$$\mathrm{Rm} = \frac{Lv}{\eta} \qquad (18.2)$$

where the magnetic diffusivity $\eta = c^2/4\pi\sigma$ and σ is the electrical conductivity. It's worth noting, here, that we can "uncurl" (18.1), to get

$$\mathbf{E} = \frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{\eta}{c} \nabla \times \mathbf{B} \qquad (18.3)$$

– and remember from Maxwell that $\partial \mathbf{B}/\partial t = -c\nabla \times \mathbf{E}$; so there is also an $\mathbf{E}$ field in the system when we have either advection ($\mathbf{v} \times \mathbf{B}$) or gradients ($\nabla \times \mathbf{B}$).

Now, envision a region of adjoining, magnetically isolated regions, as in Figure 18.1, and think about how it might evolve.

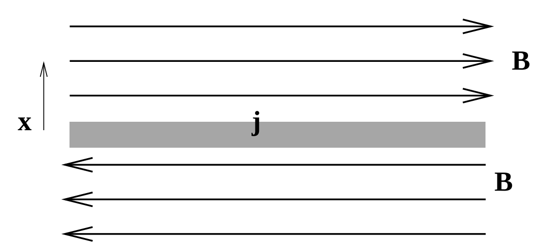

Figure 18.1. The setting for a reconnection event – the oppositely directed $\mathbf{B}$ field separates plasma in the regions above and below the current sheet ($\mathbf{j}$, located at $x \simeq 0$).

1. RESISTIVE LIMIT: FIELD DECAY ON AXIS

First, consider the highly resistive case, when $\mathrm{Rm} \ll 1$. We have simple diffusion,

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} \tag{18.4}$$

As we saw earlier, we expect the field close to the $x = 0$ line to decay, and the current sheet to broaden, with time (the first is also described as $\mathbf{B}$ lines "diffusing through the plasma"), This basic equation has well-known solutions; I'm storing one in the Appendix to this chapter. But we don't need the full solution to guesstimate the timescale for this process. If the x-extent of the system is initially $\sim L$, then from (18.4) we can directly estimate the *diffusion time*:

$$\tau_{diff} \simeq \frac{L^2}{\eta} \tag{18.5}$$

This limit destroys magnetic energy, and heats the plasma – but it doesn't, by itself, change the magnetic topology.

2. ADVECTION LIMIT: FIELD GROWTH ON AXIS

Alternatively, consider a highly conductive case, when $\mathrm{Rm} \gg 1$. Now, the first term in (18.1) dominates;

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{18.6}$$

and we talk about magnetic *flux freezing*. In the context of Figure 18.1, envision the plasma flowing towards the current sheet, with some velocity V_o. Because of flux freezing, the $\mathbf{B}$ lines will be carried with the plasma, in towards $x = 0$. Thus, $\mathbf{B}$ will grow, close to the x-axis (the field lines must bunch up as they're carried inwards). I've also put one analytic solution to this problem in the Appendix to this chapter. Once again, we can also estimate a (*dynamic*) timescale for this process, without knowing the details:

$$\tau_{dyn} \simeq \frac{L}{V_o} \tag{18.7}$$

Its also worth noting that this process can't continue forever – at some point the magnetic energy will build up enough to exert a back pressure and shut down the flow ... unless this energy is dissipated as heat which can escape the system. That means that we have to incorporate both advection and dissipation ... which leads us to reconnectoin.

But first, one more timescale. We'll find below that the inflow velocity V_o is something to be solved for, not a parameter of the problem .. and so people usually prefer to estimate an *Alfven time:*

$$\tau_A \simeq \frac{L}{v_A} \tag{18.8}$$

where v_A is evaluated at large distances from the current sheet.

We might expect that τ_A would be the best we could do to annihilate magnetic flux – simply drive the field into the dissipation region. We'll see below that things aren't this easy – that τ_A underestimates the timescales (or overestimates the reconnection rate). The inflow must be regulated by the dissipation rate ... so the final answer for the reconnection timesscale will be somewhere between τ_A and τ_{diff}.

B. Steady, 2D Reconnection

Our next task, then, is to combine advection and diffusion. When we keep both terms in (18.1), we lose the chance at simple analytic solutions. Instead, we'll work with dimensional and scaling arguments[2]

1. WHY DO FIELD LINES "BREAK"?

To understand how resistivity can "break" and "reconnect" field lines, think about the geometry in Figure 18.2. We know resistivity is important in regions of high current density - such as the central region (around OP). If we set up this geometry and waited awhile, the central magnetic field would decay as the current layer supporting them is dissipated. This would deplete the magnetic pressure in this region. The plasma above and below this region would be pushed inwards by its own pressure, bringing a in a fresh supply of field (and plasma).

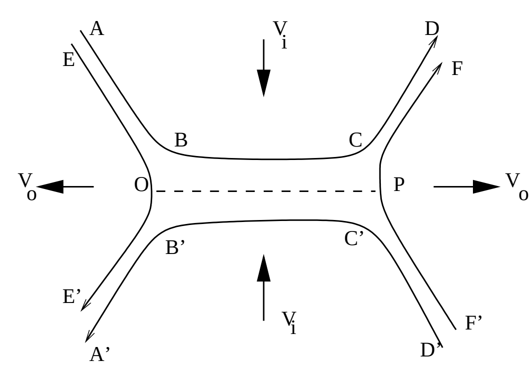

Figure 18.2. Illustrating a sinple reconnection geometry; see text for discussion. Following Choudhuri figure 15.2.

[2] This is the true state of the field, folks .. it's either these scaling arguments, or numerical simulations.

Now, look at this in more detail (I'm following Choudhuri's discussion here). The field lines ABCD and A'B'C'D' move inwards, with velocity v_{in}. Eventually the BC and B'C' parts of the field lines decay away. The AB part of the field line is moved to EO, and the A'B' part of that field line is moved to E'O. Thus, these "fragments" of two original field lines now make up one new field line, EOE'. And similarly, the parts CD and C'D' eventually make up a new field line, FPF'. Thus, "cutting and pasting" of field lines (otherwise known as reconnection) takes place in the central region. And, of course, there must be plasma flow away from the region – sideways in this cartoon – to conserve mass.

2. SWEET-PARKER MODEL

We can be more quantitative about this geometry, and find simple scaling laws to describe this situation. The model I'm describing here is *Sweet-Parker reconnection*, and is illustrated in Figure 18.2.

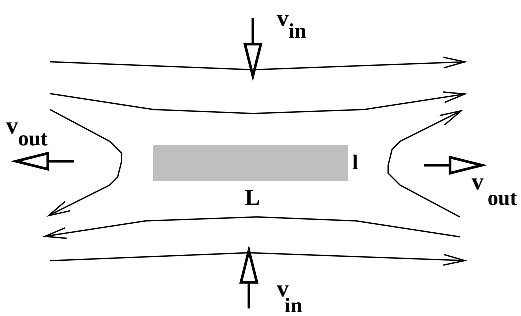

Figure 18.3. Geometry of Sweet-Parker reconnection. The current sheet (grey shaded area) has transverse width l and lateral extent L The input velocity is v_{in} and the output velocity is v_{out}. The induction $\mathbf{E} \propto \mathbf{v} \times \mathbf{B}$ is normal to the system (into/out of the paper), as is the current in the current sheet. Quantities far away from the current sheet are labelled with subscript o. Following Cowley figure 5.5.

Our analysis is strictly two-dimensional and steady state. I'll assume the flow is incompressible – that it stays at constant density (which turns out to be a good approximation if v_{in} and v_{out} are both subsonic. There are three important steps here.

• First, mass conservation requires

$$v_{in} L = v_{out} l \tag{18.9}$$

From this, directly, conservation of magnetic flux gives

$$B_o v_{in} = B_{out} v_{out} \tag{18.10}$$

Thus: the outflow speed must be significantly faster than the inflow speed (because we're assuming $l \ll L$). The outflow field is much smaller than the inflow; we're annihilating magnetic field and energy (where does it go?).

• Next, use the induction equation. In a steady state, it is $\nabla \times (\mathbf{v} \times \mathbf{B}) = \eta \nabla^2 \mathbf{B}$, which gives (by dimensional/scaling analysis)n

$$\frac{v_{in} B}{l} \simeq \frac{\eta B}{l^2} \; ; \quad v_{in} \simeq \frac{\eta}{l} \tag{18.11}$$

This is saus that we can keep the diffusion region steady if the rate at which flux is brought in is equal to the rate at which it is annihilated. We can also note that there must be an $\mathbf{E}$ field, as shown in the figure, to maintain the current sheet. From Maxwell, we get

$$\mathbf{j} = \sigma \mathbf{E} \; ; \;\; \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \Leftrightarrow \frac{B}{l} \simeq \frac{4\pi}{c} \sigma E \tag{18.12}$$

where σ is again the electrical conductivity.

• Thirdly, consider force balances. In the vertical direction, we note that $B \to 0$ at the center of the current sheet, and that $p \to p_{max}$ there (its maximum value). Pressure balance in this direction therefore requires

$$\frac{B_o^2}{8\pi} \simeq p_{max} - p_o \tag{18.13}$$

Along and within the current sheet (call that the $\hat{\mathbf{y}}$ direction), there is no $\mathbf{v} \times \mathbf{B}$ force, so only the pressure gradient accelerates the flow. We have then,

$$\rho v_y \frac{\partial v_y}{\partial y} \simeq -\frac{\partial p}{\partial y} \; ; \qquad \frac{v_{out}^2}{2} \simeq p_{max} - p_o \tag{18.14}$$

Thus, the outflow speed must be

$$v_{out}^2 \simeq v_A^2 = \frac{B_o^2}{4\pi\rho} \tag{18.15}$$

OK: we can combine these three parts to get our main results, namely, the inflow velocity ahd thickness of the dissipation layer:

$$v_{in}^2 \simeq \frac{v_A \eta}{L} \; ; \qquad l^2 \simeq \frac{\eta L}{v_A} \tag{18.16}$$

To connect with the literature, we can also rewrite our results in terms of Rm:

$$v_{in}^2 = \frac{v_{Ai}^2}{\mathrm{Rm}} \; ; \;\; B_{out}^2 = \frac{B_o^2}{\mathrm{Rm}} \; ; \;\; l^2 = \frac{L^2}{\mathrm{Rm}} \tag{18.17}$$

Thus, when $\mathrm{Rm} \ll 1$ – which we expect for the highly conductive plasmas in our examples – reconnection is indeed slow.

3. ENERGETICS

Short notes here: where does the energy go? When you work through the energetics, you'll find that the incoming kinetic energy is small (because the inflow speed is slow); most of the incoming energy is electromagnetic (Poynting flux). But also, magnetic energy is dissipated in the current sheet; this drives a plasma outflow. One can show that the outgoing kinetic energy (in the plasma flow) is equal to half of the incoming EM energy (question: where does the other half go?)

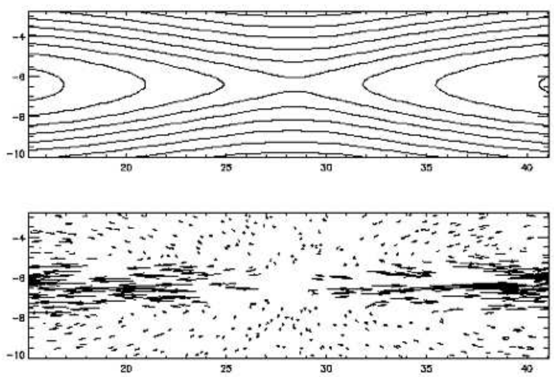

Figure 18.4. Numerical solution for Sweet-Parker-type reconnection. This simulation (presumably) started with a field geometry like that in Figure 18.1, and continued until it reached a steady state, as shown. The upper figure shows magnetic field lines; the lower one shows velocity vectors. You can see that a slow flow in from the top and bottom has changed the field topology and driven "jets" of plasma out the sides. From James Drake, Maryland.

4. TIMESCALES AND RATES

This analysis gives us, finally, the *spontaneous reconnection rate*; the rate of slow inflow that allows thinge to go steadily.

Many authors describe the v_{in} solution as a "reconnection rate", despite the imprecision of this term. Or, they use its Alfven Mach number (defined in terms of the asymptotic B_o) as a rate:

$$\mathcal{M}_{A,in} = \frac{v_{in}}{v_{Ao}} \simeq \frac{1}{(\mathrm{Rm})^{1/2}} \tag{18.18}$$

We might better talk about a "reconnection time", as a geometric mean of the (slow) diffusion time and the (much faster) dynamic time, $\tau_{dyn} = L/v_{Ai}$:

$$\tau_{rec} \simeq \frac{L}{v_{in}} \simeq (\tau_{diff}\tau_{dyn})^{1/2} \tag{18.19}$$

Clearly this type of reconnection is "medium fast": $\tau_{dyn} < \tau_{rec} < \tau_{diss}$. (Or, in terms of Alfven mach numberes, any "rate" $\sim 1/(\mathrm{Rm})^{1/2}$ is called "slow reconnection".)

Thus, Sweet-Parker reconnection is faster than just ohmic decay; but observed reconnection events proceed much faster than this simple model predicts. People have, therefore, spent a lot of time trying to invent faster versions of this model. Some are as follows ..

C. Speed up the reconnection?

I'm just putting brief comments here; this section would be far too long if I attempted a full review of the arguments.

1. PETSCHEK RECONNECTION

This model has gotten a lot of attention in the literature, but does not seem to be supported by recent work. Several people approached the slowness problem in the Sweet-Parker model by changing assumptions about the geometry; the Petschek model (Figure 18.35) is the most generally discussed of these models. The point of the Petschek model, and its relatives, is to keep the lateral extend of the diffusion region small: if the reconnection region occupies only a small fraction of the system size, then the scaling arguments above give an inflow rate faster than what the Sweet-Parker model predicts.

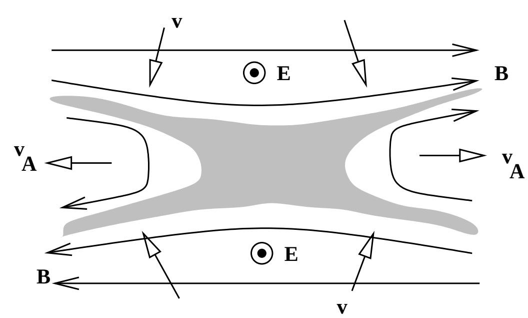

Figure 18.5. Sketch of the field and flow configuration in the Petschek reconnection model. A small central diffusion region, of scale L^*, bifurcates into two, standing-wave current sheets (standing slow-mode shocks) in the downstream flow; again shown shaded. Following Cowley Figure 5.6.

In Petschek's original model,[3] he argued that two slow-mode standing shocks exist in the outflow region, and they determine the lateral size, L^*. Similar, more general, solutions to this were also given by Sonnerup (1970). This class of solutions is often quoted in the literature as the answer to the slowness problem. These

[3] Note that this is not A. Petschek who used to teach at Tech; I believe it is a relative.

solutions assume that the diffusion region will adjust itself to respond to the external boundary (driving) conditions, in doing so making $L^* \sim l$ and thus making the reconnection rate nearly independent of η.

Do Petschek-type mechanisms work? It looks at present as if the Petschek mechanism is not obtained in real situations. Several numerical simulations (*e.g.*, Biskamp 1986, Ugai 1995) have looked for steady reconnection solutions, and found that simple, two-dimensional systems will reach the Sweet-Parker solution for low inflow rates. At higher rates, however, the simulations find a transition to unsteady behavior, rather than a transition to the Petschek mode. In addition, recent lab work which was configured explicitly to measure reconnection physics (*cf.*review by Kulsrud, 1998) seems to support the Sweet-Parker model.

2. COMPRESSIBLE FLOW

Another suggestion is that the plasma in the dissipation region (DR) is compressed. If this is the case, then (18.9) is replaced by $v_{in}\rho_o L \simeq v_{out}\rho_{DR}v_{out}$. This will increase v_{in} relative to v_{out} – its a plausible idea, but I'm not aware of any lab tests yet. This will be investigated in the homework.

3. ANOMALOUS DIFFUSION

This is my personal favorite, and is a good example of how plasma microphysics can be important in macroscopic situations. That is: the electrical resistivity σ, which sets the size of the η term, is traditionally calculated assuming Coulomb collisions are the mechanism for energy transfer and dissipation. (Check the appendix of Chapter 1 for Coulomb scattering, and the back of Chapter 13 for conductivity discussions). The critical point is that electrical conductivity is regulated by the collision rate of the current-carrying electrons:

$$\sigma = \frac{j}{E} = \frac{nev_D}{E} \simeq \frac{ne^2}{m}\tau_{coll} \tag{18.20}$$

where τ_{coll} is the electron collision time.

Now, Coulomb collisions are the "hands-off" particle-particle collisions that result from the long-range nature of the electrical force. They must happen in any ionized plasma – but they are slow (at least in low-density plasmas, such as space or the lab), and thus not very effective at dissipating a current. Are Coulomb collisions the only dissipation mechanism? The answer is, almost certainly not. If the plasma is turbulent on microscopic scales (which is very likely to be the case), and if the turbulence involves charge separation (ditto), we expect the plasma to contain fluctuating electric fields (again on microscopic scales; they won't affect the macroscopic dynamics that we're studying in this course).

> Plasma physics details: the most common origin of microturbulence is *streaming* of charges relative to the background plasma. If the streaming speed is high enough – typically greater than the plasma thermal speed, $v_{th} \propto \sqrt{kT/m}$ – the system is unstable, and some of the streaming kinetic energy is turned into microturbulence. The most common way to set up relative streaming is with a high current density. As above, $j = nev_D$; a higher j leads to a higher drift velocity v_D, and when $v_D > v_{th}$ streaming instabilities occur. But the current density is controlled by external conditions – for instance by the scale over which $\mathbf{B}$ changes significantly, in the simple reconnection models. Thus, thin current sheets ⇒ high current density ⇒ microturbulence.

Once microturtubulence is created, stochastic electric fields will scatter the plasma particles; if the plasma turbulence is strong enough (and very weak levels will do) this *turbulent scattering* will be much more important than single-particle interactions. Because this is a non-linear process (at what amplitude does the turbulence saturate?), we can't write down collision times from first principles. One common approach (way to get around this) argues that we can get an upper limit to the resistivity (lower limit to τ_{coll} is to set $\tau_{coll} = 2\pi/\omega_p$, if $\omega_p^2 = 4\pi ne^2/m$ is the square of the *plasma frequency* (in cgs units). This is used in (18.20) to estimate *anomalous conductivity* or *anomalous resistivity*.

D. Other approaches

Reconnection is a very active field these days. I'm just putting a few short comments here; once again, these notes would be very long if I tried to review the entire field.

1. SPONTANEOUS RECONNECTION

The simple models assumed a steady state, with some mass inflow to balance the flux annihilation. Thus they are "driven" in a general sense. But the plasma will find its own way; oppositely directed magnetic fields across a thin current sheet will spontaneously reconnect. This is called a *tearing* instability; and typically results in formation of *magnetic islands* in the current sheet. Figure 18.6 illustrates this behavior.

This instability can be studied analytically; the math is long and horrendous, and I'll put it in a later chapter. It can also be studied with numerical simulations; an example of that will also be in that later chapter.

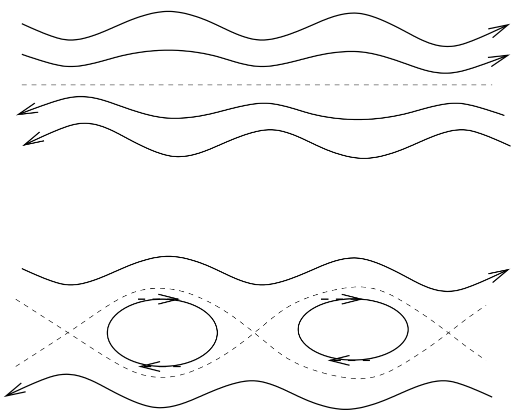

Figure 18.6. Effect of a perpendicular plasma displacement in a neutral sheet, leading to field compression for the ideal case ($\eta = 0$; top figure) and to field disruption and island formation for the resistive case ($\eta \neq 0$; lower figure). Following Biskamp Figure 4.4.

2. DRIVEN RECONNECTION

This is a different approach. It's worth remembering that the 2D models above are all "passive": put two misaligned $\mathbf{B}$ fields together and wait to see how quickly they reconnect. But nature does not always work this way. One can envision a situation in which the two anti-parallel magnetic structures are *driven together*, by large-scale flows in the system (one example of this is MHD turbulence, in which different parts of the plasma move in random directions, at a speed set by the energetics of the turbulence). In this case one (this one at least) expects that the inflow speed (v_{in}) in our notation above) will be set by the large-scale flow (say the turbulent speed). How can the reconnection site adjust to this? Some authors (*e.g.* Parker) suggest that the internal structure of the reconnection layer – its thickness, density or resistivity (set by microscale turbulence therein) – will adjust as necessary. I personally tend to agree with them.

3. NON-STEADY RECONNECTION

Here's another personal impression: who says steady-state reconnection is relevant to any natural situation? That is: the arguments above show that steady reconnection models must be forced, sometimes rather severely, to connect with what we think is occuring in nature. But examples of non-steady reconnection events are easy to find:

• Reconnection in solar flares. Flares are seriously transient events; the large amount of energy released is believed to be due to very fast annihilation of magnetic field, in a reconnection event. Flares have motivated much of the work on steady-state models; however as an outsider I suspect time-dependent, patchy, localized reconnection must be taking place.

• Reconnection at the magnetopause, where the solar wind hits the earth's magnetic field. Spacecraft observations suggest this is very patchy, localized and time-dependent. Picture, for instance, magnetic flux tubes being carried along in the solar wind; and let one of them impact the magnetopause, which also has its field bunched into ropes. This process can allow solar wind plasma, and field, to penetrate into the magnetosphere.

There are many numerical simulations of dynamic reconnection in the literature; I'll bring some to class.

4. THREE-DIMENSIONAL RECONNECTION

Another observation: only rarely can a reconnection event be well described by a two-dimensional analysis. My last example in fact assumed this – because the intersection of two magnetic flux ropes is clearly a three-dimensional process. Going to 3D is challenging, and work is only starting here (helped significantly by recent advances in resistive MHD codes). If I can I'll bring some recent images to class.

References

Some of this is "just from me", from my reading in the field. I've taken the basic Sweet-Parker and Petschek material from Priest; and the two formal solutions in the Appendix from Priest & Forbes.

E. Appendix

1. DIFFUSION-ONLY SOLUTION

Refer back to Figure 18.1, so that we have a 1D problem, $\mathbf{B} = (0, B(x,t), 0)$; and take the figure as an initial condition

$$B = B_o, x > 0\,; B = -B_o, x < 0 \qquad (18.21)$$

The time evolution of the system is governed by

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2} \qquad (18.22)$$

There are scads of different ways to solve this equation. One uses a Green function:

$$B(x,t) = \int G(x-x',t)B(x',0)dx' \tag{18.23}$$

and for this initial condition, the Green function is

$$G(x-x',t) = \frac{1}{(4\pi\eta t)^{1/2}} e^{-(x-x')^2/4\eta t} \tag{18.24}$$

From this we get the desired solution, in terms of an error function:

$$\begin{aligned} B(x,t) &= \frac{2B_o}{\sqrt{\pi}} \operatorname{erf}\left(\frac{x}{\sqrt{4\eta t}}\right) \\ &= \frac{2B_o}{\sqrt{\pi}} \int_0^{x/\sqrt{4\eta t}} e^{-u^2} du \end{aligned} \tag{18.25}$$

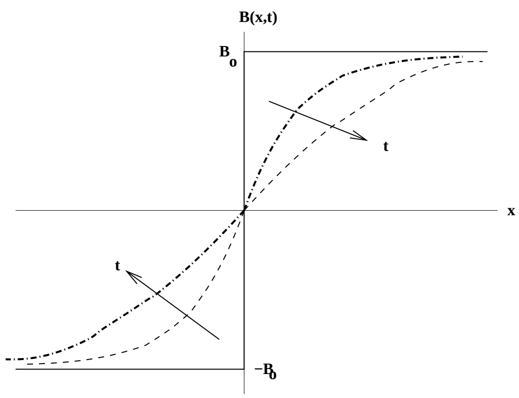

Figure 18.7. Solution for the magnetic field as a function of time, "diffusing" away from a current layer at $x = 0$. The arrow shows the direction of increasing time; solution details are in the text. Following Priest & Forbes figure 3.1.

This has the expected behavior, as illustrated in Figure 18.7: the B field decays with time, near the current sheet, which itself spreads with time. One can also show that the total current stays constant:

$$j = \frac{c}{4\pi}\frac{\partial B}{\partial x} \;; \quad J = \int j dx = \frac{cB_o}{2\pi} \tag{18.26}$$

and that the magnetic energy decays with time:

$$\begin{aligned} \frac{du_B}{dt} &= \int \frac{B}{4\pi}\frac{\partial B}{\partial t} dx = \int \frac{B}{4\pi}\eta\frac{\partial^2 B}{\partial x^2} dx \\ &= -\frac{\eta}{4\pi}\int\left(\frac{\partial B}{\partial x}\right)^2 dx \\ &= -\frac{\eta}{4\pi}\int\left(\frac{4\pi}{c}j\right)^2 dx = -\int\frac{j^2}{\sigma}dx \end{aligned} \tag{18.27}$$

(where we've integrated by parts in the second line, and used basic Maxwell in the third).

2. ADVECTION-ONLY SOLUTION

Refer again to Figure 18.1, but this time specify a velocity field which describes inflow towards $x = 0$, but also allows flow out the sides (to conserve mass):

$$v_x = -V_o\frac{x}{a} \;; \quad v_y = V_o\frac{y}{a} \tag{18.28}$$

This satisfies $\nabla \cdot \mathbf{v} = 0$ (incompressible); and has streamlines xy = constant.[4] As an initial condition, pick a $\mathbf{B}$ field which increases going away from $x = 0$, and changes sign at $x = 0$:

$$B(x, t=0) = B_o \cos\frac{x}{a} \tag{18.29}$$

Figure 18.8 shows this geometry. If we ignore diffusion, our governing equation is

$$\frac{\partial B}{\partial t} - V_o\frac{x}{a}\frac{\partial B}{\partial x} = \frac{V_o}{a}B \tag{18.30}$$

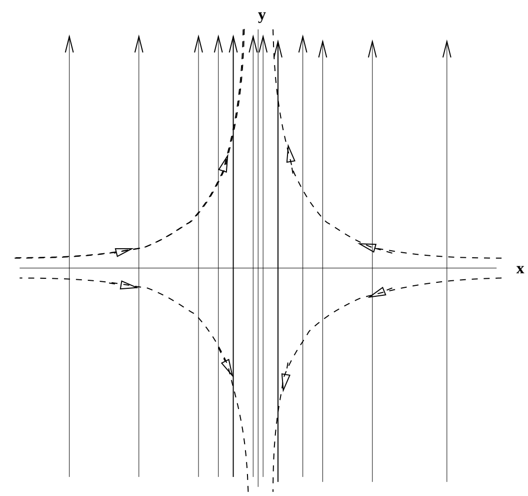

Figure 18.8. Solution for the magnetic field as a function of time, being advected into a current layer at $x = 0$. The dotted lines are velocity streamilines; the magnetic field builds up close to $x = 0$ as time goes on. Solution details are in the text. Following Priest & Forbes figure 3.3.

Priest and Forbes present a cute solution to this. Consider the characteristic lines (in the (x,t) plane)

$$x(t) = x_* e^{-V_o t/a} \tag{18.31}$$

(clearly x_* is the initial value of x, on a given velocity line, at $t = 0$). Along these lines, B obeys

$$\begin{aligned} \frac{dB}{dt} &= \frac{\partial B}{\partial t} - \frac{dx}{dt}\frac{\partial B}{\partial x} \\ &= \frac{\partial B}{\partial t} - V_o\frac{x}{a}\frac{\partial B}{\partial x} = \frac{V_o}{a}B \end{aligned} \tag{18.32}$$

[4] Check back to Chapter 1 for the definition of streamlines, and how to find them from the velocity field.

(the first line is just chain rule, and the second line notices connects back to eqn 18.30). But now it's easy: along the characteristic lines, (18.31), the solution is

$$B(x,t) = B_o e^{V_o t/a} \cos\left(\frac{x_*}{a}\right) \tag{18.33}$$

All we have left to do is write this in terms of (x, t) everywhere:

$$B(x,t) = B_o e^{V_o t/a} \cos\left(\frac{x}{a} e^{V_o t/a}\right) \tag{18.34}$$

This shows that the field accumulates – grows in strength – near the $x = 0$ axis; in fact it concentrates there (closer and closer to $x = 0$ as time goes on.

However, this clearly isn't a physical solution, because if B grows exponentially, magnetic pressure will react back on the flow.