19. FLUID INSTABILITIES

Fluids, magnetized or not, are subject to a wide variety of instabilities. Many so-called equlibrium configurations, that is steady-state solutions of the basic equations (or configurations that we might set up in the lab), are violently unstable. That is, once set up, they rapidly evolve away from their initial state. Sometimes these instabilities are destructive (such as current-driven instabilities of plasma pinches), and sometimes they move the system to another quasi-steady state (such as convection in the atmosphere or on the sun).

In this chapter we consider a few of these, both hydrodynamic and magnetohydrodynamic. In each case, we are interested in a criterion for, and a description of, instability. When – for what initial configurations – is a system unstable? How fast, and on what scales, does the instability grow? The answers to these questions require some lengthy math. We will see two different approaches. One is *modal analysis*, in which a small perturbation is studies in terms of its Fourier components. In this type of analysis, the system is determined to be unstable if the frequency of the perturbation is found to have imaginary components. Another method is *the energy method*, in which one determines whether the potential energy of an equilibrium system is a maximum (unstable), or a minimum (stable).

Often the physical insight on why the instability goes, arrives only after the analysis has been gone through. In these notes, I will try to focus on the physics driving each instability, and present only a summary of the mathematical analysis, rather than reproducing every step (with credit to each author who presents it in more detail).

A. Buoyancy and Thermal Convection

To motivate things, recall one example we've already seen. In chapter 6 we discussed convective stability, and in chapter 14 we talked about the effects of a magnetic field. Let's revisit this briefly, aiming at the physics, but with an eye to the more formal stability analysis about to come.

In §6.2, we considered a small vertical displacement of a parcel of fluid (air, say) in a static atmosphere. We expect, physically, that this parcel will be *stable* against the displacement if it has a higher density than its surroundings, once displaced; it will just bob up and down. The frequency of that motion is the Brunt-Vaisala frequency, N (as derived in §6.2). On the other hand, if the blob when displaced has a lower density than its surroundings, it will be *unstable*, and continue to rise. In this case, *convection* will develop.

To find the conditions under which this instability occurs, we argued that the rising blob is adiabatic, and also stays in pressure balance with the outside atmosphere. We found, somewhat heuristically, that the stability of the blob depends on how rapidly the density of the atmosphere drops compared to the adiabatic case. If we didn't have this physical argument in our head, we could have done it more formally, by assuming the blob oscillates as e^{iNt}, and exploring what N is. Check the result (6.15): the stability criterion is in this equation. If $N^2 > 0$, the blob is stable; while if $N^2 < 0$, the displacement grows exponentially and the system is unstable. But the formal result, (6.15), shows that stability depends on the sign of $dT/dz - (dT/dz)_{ad}$.

Thus: stability or instability is determined by the temperature structure of the atmosphere – which is determined by external conditions (how we set the problem up). In this case we could make a physical argument to lead us to the stability criterion; and a formal analysis found the same result. In more complex cases, the exact form of the physical argument is not always evident ... but the mathematical approach, analogous to the e^{iNt} method here, will lead us to the answer.

In the rest of this chapter I'll treat two well-known fluid instabilities analytically, with and without magnetic fields. Hold on to your hats ...

B. The Rayleigh Taylor Instability

1. THE PHYSICS

This instability arises when a heavy fluid is supported on top of a light one, in a gravitational field. This is clearly unfavorable energetically; if the two fluids can change places, the system will have a lower potential energy. This is just what the RT instability does; any tiny perturbation of the interface between the two fluids, allows "fingers" to develop – in which each fluid penetrates into the other.

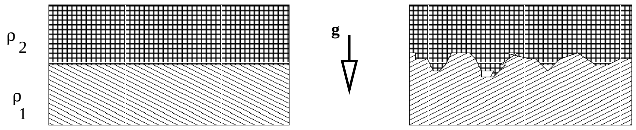

Figure 19.1. Development of the Rayleigh-Taylor instability. The fluid on top has higher density, and gravity points downward. Mixing due to the instability produces fingers that eventually invert the density distribution.

We will work this in terms of a fluid interface; but any density gradient inverted compared to the local gravity gives the same results. In addition, the acceleration does not need to come from gravity; a decleration of a light fluid into a heavy one (as in an explosion in a dense atmosphere) is also subject to this instability.

2. THE MATH

Our basic approach here is modal analysis. That is: first we linearize, assuming small perturbations, so that we can drop terms which are second order in these perturbations. We have done this before – in deriving sound waves and MHD waves, for instance. Our goal is to find the time evolution of these perturbations: do they grow with time (which is an instability), or simply oscillate (which says we have wave solutions, but not instabilities). In order to do this, we do a *modal analysis*. We assume that any perturbation can be treated as a sum of Fourier components (which is legal if the defining equations are linear in the perturbations – right??). We can then focus on the evolution of one such Fourier component – for instance (19.2), below.

My analysis here follows Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. He works through a modal analysis, and we will pick up on some of his results. Recall our formal derivation of sound waves, in Chapter 4: we carried through a linear analysis, expanding the variables $\mathbf{v}, p, \rho$ in (unperturbed) + (small perturbation) terms. Our basic equations are those of mass conservation, incompressibility of the unperturbed state, and momentum conservation. Geometry: take $\mathbf{g} = g\hat{\mathbf{z}}$; assume the unperturbed state had $\mathbf{v} = 0$, and only has gradients on the $\hat{\mathbf{z}}$ direction. and let the perturbed velocity components be $\delta\mathbf{v} = (u, v, w)$. The linearized equations – for the perturbations $\delta\rho, \delta p$, keeping only lowest order "small" terms, are:

$$\begin{aligned}
\rho\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\delta p \\
\rho\frac{\partial v}{\partial t} &= -\frac{\partial}{\partial y}\delta p \\
\rho\frac{\partial w}{\partial t} &= -\frac{\partial}{\partial z}\delta p - g\delta\rho \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &+ \frac{\partial w}{\partial z} = 0 \\
\frac{\partial}{\partial t}\delta\rho &= -w\frac{d\rho}{dz}
\end{aligned} \tag{19.1}$$

Now, assume each perturbed quantity has the form (a "mode")

$$\delta f(x, y, z, t) = \delta f(z) e^{i(k_x x + k_y y + \omega t)} \tag{19.2}$$

We will generally treat $\mathbf{k}$ as given, and want to find the nature of ω. In particular: if ω is real, (19.2) simply represents waves or oscillations – so that the system is stable. However if ω has a negative imaginary part, then (**??**) represents a perturbation with grows exponentially with time: the system is *unstable*. Let $k^2 = k_x^2 + k_y^2$, and denote z-derivatives by $Df = df/dz$ (In this and the next section only; Chandra's notation). These equations then become

$$\begin{aligned}
ik_x\delta p &= -i\omega\rho u \\
ik_y\delta p &= -i\omega\rho v \\
D\delta p &= -i\omega\rho w - g\delta\rho \\
ik_x u + ik_y v &+ Dw = 0 \\
i\omega\delta\rho &= wD\rho
\end{aligned} \tag{19.3}$$

These can be used to show

$$k^2\delta p = -i\omega\rho Dw \tag{19.4}$$

and

$$D\delta p = -i\omega\rho w + \frac{g}{i\omega}wD\rho \tag{19.5}$$

These two now combine – eliminating δp between them – as

$$D(\rho Dw) = \frac{gk^2}{\omega^2}wD\rho + \rho wk^2 \tag{19.6}$$

But now, this is an equation for $w(\omega, k)$, because $D\rho$ is specified in the problem. Picking $w = 0$ on the (distant) boundaries of the system, this becomes a Sturmian characteristic value problem for ω^2. One can show, then, that $\omega^2 > 0$ if $D\rho < 0$ everywhere; and conversely, $\omega^2 < 0$ if $D\rho > 0$ somewhere in the fluid. That is: a density structure which decreases with $\hat{\mathbf{z}}$ is *stable*; while a density structure which increases with $\hat{\mathbf{z}}$ is *unstable*.

The next step is to find out the growth rates, and whether all wavevectors are unstable. To do this, we simplify to the case of two uniform fluids, separated by a horizontal interface at $z = 0$. Now, away from this boundary, (19.6) becomes

$$D^2 w = k^2 w \tag{19.7}$$

which has simple exponential solutions. Picking the ones which stay well-behaved at $z \to \pm\infty$, we have away from the boundary

$$\begin{aligned}
w(z) &= Ae^{-kz}\ ; \qquad z > 0 \\
w(z) &= Ae^{kz}\ ; \qquad z < 0
\end{aligned} \tag{19.8}$$

To connect solutions through the boundary, we need jump conditions. Define

$$\Delta_s f = f(z_s + 0) - f(z_s - 0)$$

that is, the jump in f from the below some surface z_s to above that surface. Here $z_s = 0$. Applied to (**??**), we find the jumps in p and ρ are related by

$$\Delta_o \delta p = \frac{g}{\omega} w_s \Delta_o \rho \tag{19.9}$$

Manipulation of the system (**??**) also gives

$$k^2 \Delta_o \delta p = -\Delta_o \left(-i\omega\rho\right) \tag{19.10}$$

Finally, these combine to give

$$\Delta_o \left(\rho D w\right) = \frac{k^2}{\omega^2} g w \Delta_o \rho \tag{19.11}$$

But in this system, $\Delta_o \rho = \rho_2 - \rho_1$ (upper = lower values); and $Dw = \pm k w$ (- for upper, + for lower). The result (19.11) thus becomes

$$\omega^2 = -gk \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) \tag{19.12}$$

This is our result: $\omega^2 < 0$ – the system is unstable – if $\rho_2 > \rho_1$. This recovers what we guess from potential energy arguments, at the start. We also find that *all* wavenumbers are unstable (that is, have $\omega^2 < 0$), with a growth rate $|\omega| \propto \sqrt{k}$. This last result is true for small perturbations; however when the instability starts to grow, larger-scales dominate the system.

This analysis could also have included surface tension at the interface: when included, one finds that surface tension stabilitizies high wavenumbers (small spatial scales).

3. THE MAGNETIZED CASE

What happens if we add a magnetic field? We expect little effect from a vertical field, as it doesn't hamper vertical fluid motions. This turns out to be the case – the stability conditions are not affected. It does have some effect on the growth rate, however (note that a vertical perturbed velocity must also involve horizontal, cross-field flows – by the continuity equation). The more interesting case is a horizontal magnetic field: we expect this to affect the stability as well as the growth rates, and it does.

In this section I still follow Chandra. Consider a uniform $\mathbf{B} = B\hat{\mathbf{x}}$, with perturbed field

$$\delta \mathbf{B} = (b_x, b_y, b_z) = \frac{k_x}{\omega} B \mathbf{v} \tag{19.13}$$

(Why is it proportional to the perturbed velocity?) The perturbed momentum equations become

$$\begin{aligned} i\rho\omega u &= -ik_x \delta p \\ i\rho\omega v &= \frac{B}{4\pi}\left(ik_x b_y - ik_y b_x\right) - ik_y \delta p \\ i\rho\omega w &= \frac{B}{4\pi}\left(ik_x b_z - iD b_x\right) - ik_y \delta p + \frac{g}{i\omega} w D\rho \end{aligned} \tag{19.14}$$

Now: using (19.11) to convert from $\mathbf{b}$ components to $\mathbf{v}$ components, we can show

$$\left(i\rho\omega + \frac{k_x^2}{i\omega}\frac{B^2}{4\pi}\right)\left(ik_x v - ik_y u\right) = 0 \tag{19.15}$$

which requires $k_x v = k_y u$. Using this result, and further algebra, we find

$$i\rho\omega D w = -k^2 \delta p \tag{19.16}$$

and

$$i\rho\omega w - \frac{B^2 k_x^2}{4\pi k^2 i\omega}\left(D^2 - k^2\right) w = -D\delta p + \frac{g}{i\omega} w D\rho \tag{19.17}$$

And now, we can look for the new results. First, combine (19.16) and (19.17) to get

$$D\left(\rho D w\right) - \rho w k^2 - \frac{B^2 k_x^2}{4\pi k^2 \omega^2}\left(D^2 - k^2\right) w = \frac{g k^2}{\omega^2} w D\rho \tag{19.18}$$

which is the generalization of (19.4) for the non-magnetic case.

From this, first, we note that if $k_x = 0$, which corresponds to propagation perpendicular to the field, (**??**) just reduces to (19.6): cross-field perturbations have no effect on the RT instability. The ones with $k_x \neq 0$, however, do change the results. We can repeat the non-magnetized analysis . . . the jump condition (19.11) generalizes to

$$\Delta_o \left(\rho D w\right) - \frac{B^2 k_x^2}{4\pi\omega^2}\Delta_o(Dw) = \frac{k^2}{\omega^2} g w \Delta_o \rho \tag{19.19}$$

Again taking the case of two uniform fluids separated by an interface, the result (19.12) becomes

$$\omega^2 = -gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) + \frac{2B^2 k_x^2}{4\pi\left(\rho_2 + \rho_1\right)} \tag{19.20}$$

Thus: a horizontal magnetic field stabilizes high frequency perturbations. Instability ($\omega^2 < 0$) requires

$$gk\left(\rho_2 - \rho_1\right) > \frac{2B^2 k_x^2}{4\pi} \tag{19.21}$$

Finally, we note that this analysis assumed an incompressible fluid. That simplifies the algebra but isn't required for the physics. Extending to the compressible case (as done by Shivamoggi, *Theory of Hydromagnetic Stability*), one finds that compressibilty has a stabilizing effect, in that it *reduces* the growth rates in both the magnetized and unmagnetized cases, compared to their incompressible analogs.

C. The Kelvin Helmholtz Instability

Well, that was so much fun, let's do it again. The other well-known instability of an interface is the Kelvin Helmoltz Instability.

1. THE PHYSICS

This instability arises at a velocity shear. Consider two fluids, or two pieces of the same fluid, in relative motion. For instance, think of a horizontal interface; fluid below is at rest, and fluid above is moving parallel to the interface. (This might describe a situation in the atmosphere, with high-velocity winds going past a low-velocity cloud layer). When the interface is displaced upwards, it forces the overlying flow to deviate over the perturbation. This means the pressure there drops (Bernoulli, remember?). The boundary thus feels a lift. But of course the situation is mirror symmetric across the interface; the downward displaced boundary feels a downward "lift". As the boundary follows these perturbations, it is sheared by the flow into which it penetrates, and rollw up into a vortex sheet.

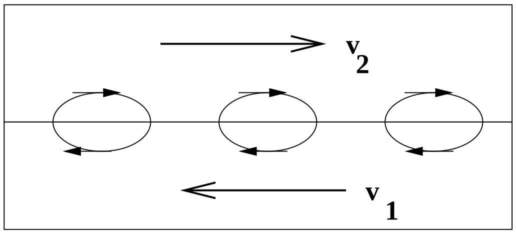

Figure 19.2. The origin of the Kelvin-Helmholtz instability. Shear at an interface between two fluid layers in relative motion generates a vortex sheet; pressure differences (think of the Bernoulli effect) lead to instability, and eventual mixing of the two fluids.

2. THE MATH

We again follow Chandrasekhar. The methods are similar to those presented for the RT instability, but the geometry is slightly more complicated. We add a horizontal flow $\mathbf{U} = u\hat{\mathbf{x}}$ to the unperturbed system, but we now ignore gravity.

The basic perturbed equations are

$$\begin{aligned}
\rho\frac{\partial u}{\partial t} + \rho U\frac{\partial u}{\partial x} + \rho w\frac{dU}{dz} &= -\frac{\partial}{\partial x}\delta p \\
\rho\frac{\partial v}{\partial t} + \rho U\frac{\partial v}{\partial x} &= -\frac{\partial}{\partial y}\delta p \\
\rho\frac{\partial w}{\partial t} + \rho U\frac{\partial w}{\partial x} &= -\frac{\partial}{\partial z}\delta p \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial}{\partial t}\delta\rho + U\frac{\partial}{\partial x}\delta p &= -w\frac{d\rho}{dz}
\end{aligned} \tag{19.22}$$

We again assume the perturbation has the form, (19.2). The system (**??**) becomes

$$\begin{aligned}
ik_x\delta p &= -i\nu\rho u - \rho w DU \\
ik_y\delta p &= -i\nu\rho v \\
D\delta p &= -i\nu\rho w \\
ik_x u + ik_y v + Dw &= 0 \\
i\nu\delta\rho &= -wD\rho
\end{aligned} \tag{19.23}$$

where we have defined the "shifted" frequency, $\nu = \omega + k_x U$. From these we can find

$$i\rho\nu Dw - i\rho k_x w DU = -k^2\delta p \tag{19.24}$$

and

$$ik^2 D\delta p = \rho k^2 \nu w \tag{19.25}$$

(compare equations 19.4 and 19.5). These combine (eliminate δp) to give

$$D\left[\rho\nu Dw - \rho k_x w DU\right] - k^2\rho\nu w = 0 \tag{19.26}$$

and from this, our useful jump condition , at an interface z_s, is

$$\Delta_s\left[\rho\nu Dw - \rho k_x w DU\right] = 0 \tag{19.27}$$

(what quantities must be held constant across an interface? how did we throw out some of the terms in **??**?)

We again apply this to two uniform fluids. Above and below the boundary, we again have (19.7), as all else is constant. But now, the ratio $w/(\omega + k_x U)$ must be continuous across the boundary (why?). Thus, (19.8) is replaced by

$$\begin{aligned}
w(z) &= A\nu_2 e^{-kz}\,; \qquad z > 0 \\
w(z) &= A\nu_1 e^{kz}\,; \qquad z < 0
\end{aligned} \tag{19.28}$$

(where the subscripts 2,1 refer to (top),(bottom) flow speeds). Our characteristic equation is simpler here. Taking $\rho_1 = \rho_2 = \rho$, we can derive

$$(\omega + k_x U_2)^2 + (\omega + k_x U_1)^2 = 0 \tag{19.29}$$

But this has only complex roots (if $k_x \neq 0$). The quadratic solves to

$$\omega = -\frac{1}{2} k_x (U_1 + U_2) \pm \frac{i}{2} k_x (U_1 - U_2) \tag{19.30}$$

Thus, any such perturbation is unstable (ω has an imaginary part), no matter how small the difference $U_1 - U_2$ may be.

> As Chandrasekhar quotes Helmholtz: “Every perfect geometrically sharp edge by which a fluid flows must tear it asunder and establish a surface of separation, however slowly the rest of the fluid may move”.

Equation (19.30) shows that higher k’s grow the fastest when the perturbation is small. When it becomes finite, however, this analysis breaks down; experiment shows that a characteristic and large scale will dominate the instability.

3. THE MAGNETIZED CASE

Finally, we can again consider the effect of a magnetic field. We expect the interesting case will be when $\mathbf{B} \parallel \mathbf{U}$; we don’t expect cross-field flows to differ from the unmagnetized case.

The modal equations become

$$\begin{aligned}
ik_x \delta p &= -i\nu\rho u - \rho w DU \\
ik_y \delta p &= -i\nu\rho v + \frac{B}{4\pi}(ik_x b_y - ik_y b_x) \\
D\delta p &= -i\nu\rho w + \frac{B}{4\pi}(ik_x b_z - Db_x) \\
ik_x u + ik_y v + Dw &= 0 \\
i\nu\delta\mathbf{B} &= ik_x B\mathbf{v} + b_z D\mathbf{U} \\
i\nu\delta\rho &= -wD\rho
\end{aligned} \tag{19.31}$$

From these, after a fair bit of algebra, and again using the “shifted” frequency $\nu = \omega + k_x U$, we find (compare 19.16 and 19.17)

$$\rho\nu Dw - \rho k_x w DU = ik^2 \delta p \tag{19.32}$$

and

$$ik^2 D\delta p = \rho k^2 \nu w + k_x^2 \frac{B^2}{4\pi}\left[D\left(\frac{Dw}{\nu}\right) - \frac{k^2 w}{\nu}\right] - k_x^3 \frac{B^2}{4\pi} D\left(\frac{DU}{\nu^2} w\right) \tag{19.33}$$

These two combine as

$$D(\rho\nu w - \rho k_x w DU) + k_x^2 \frac{B^2}{4\pi}\left[D\left(\frac{Dw}{\nu}\right) - \frac{k^2 w}{\nu}\right] - k_x^3 \frac{B^2}{4\pi} D\left(\frac{DU}{\nu^2} w\right) \tag{19.34}$$

(compare 19.16). Specializing once more to two uniform fluids separated by a boundary at $z = 0$, our jump condition is

$$\Delta_o(\rho\nu Dw) = k_x^2 \frac{B^2}{4\pi} \Delta_o\left(\frac{Dw}{\nu}\right) \tag{19.35}$$

From this, going back from ν to ω, we get the characteristic equation:

$$(\omega + k_x U_2)^2 + (\omega + k_x U_1)^2 = k_x^2 \frac{2B^2}{4\pi} \tag{19.36}$$

The roots of this are

$$\omega = -\frac{1}{2} k_x (U_1 + U_2) \pm \left[k_x^2 \frac{B^2}{4\pi\rho} - \frac{k_x^2}{4}(U_1 - U_2)^2\right]^{1/2} \tag{19.37}$$

And so, finally, we have our result: a B field aligned with the flow will stabilize the KH instability *at all wavelengths* if

$$(U_1 - U_2)^2 \leq \frac{4B^2}{4\pi\rho} \qquad (19.38)$$

Finally, again, one can also do this for compressible fluids. As with RT, one finds that compressibility is stabilizing; it reduces the growth rate, in both the unmagnetized and magnetized cases, compared to our incompressible calculation here.

References

As in the text, I've followed Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* for the hydro, and Shivamoggi, *Theory of Hydromagnetic Stability*), for the MHD.