2. VISCOSITY AND LAMINAR FLOW: BASICS

Let's start by looking at some simple fluid flow solutions. We'll consider steady-state, incompressible flows. Some very simple solutions can be found when viscosity is important to the flow. Because viscosous stresses can, however, get very complicated, we'll begin with simple examples, then see the general stuff.

A. One-dimensional Laminar Flows

Think about a long, 1D channel such as in Figure 2.1. The flow is driven by a pressure gradient, $p_2 < p_1$; and we assume the velocity must go to zero at the walls (this is a *no-slip* boundary). Because the flow is incompressible – has the same density everywhere – the velocity profile must be independent of x, in order to enable mass conservation (the same flow rate across any point within the channel).

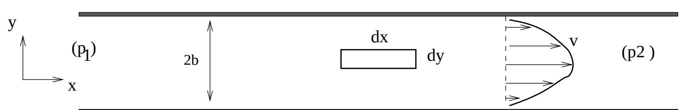

Figure 2.1. Defining the geometry for this problem. The flow is driven by a pressure gradient; $p_2 < p_1$. This driver, combined with the no-slip condition assumed at the walls and the action of viscous stresses within the flow, leads to the quadratic velocity profile shown.

Now, think about a little differential volume $dxdy$, as shown in the figure. It feels a net force to the right, due to dp/dx; what balances this force to keep the flow from accelerating? This must be viscosity: friction on the little volume due to the shear in the flow. We assume the shear stresses are linearly proportional to the velocity gradient, transverse to the surface (this is called a *Newtonian* flow). The little volume will feel friction forces on its top and bottom surfaces; the net force on the volume will be the difference between the forces on the top and bottom. If any one surface feels a force $\rho\nu(\partial v/dy)$, the net force (per length in the z direction) on our element is

$$\left[\rho\nu\left(\frac{\partial v}{\partial y}\right)_{y+dy} - \rho\nu\left(\frac{\partial v}{\partial y}\right)_{y}\right]dx = \rho\nu\frac{\partial^2 v}{dy^2}dx \tag{2.1}$$

In this last I've assumed dy is small, of course, and also that ν is independent of y. I've also written the constant of proportionality as $\rho\nu$, and the constant ν is what I like to call the *coefficient of viscosity*; I'll return to this shosrtly.

> Beware: you must be careful when working with viscosity. The definitions, symbols and even units of viscosity vary from author to author. I prefer the choice, that $\nu\nabla\mathbf{v}$ has the dimensions of a force per area per gram: and that $\rho\nu\nabla\mathbf{v}$ is a force per area. Thus, the dimensions of ν are [L]^2/[T]. Many authors (including Faber) include the density; he uses η (dimensions [M]/[L][T]) which is the same as our $\rho\nu$. Still other authors use μ where Faber uses η.

So: putting this back into vector form, we've discovered that our governing equation for this simple system should be[1]

$$\nabla p = \rho\nu\nabla^2\mathbf{v}$$

But now ... looking back to equation (1.10), our result here suggests that we should add a new term to describe the effects of viscous stresses. Taking this as true for the moment (we'll derive it more generally below), we have the basic equation governing viscous flow:

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \rho\mathbf{g} + \rho\nu\nabla^2\mathbf{v} \tag{2.2}$$

Compare this back to our first version of the momentum equation, namely (1.10), which we called Euler's equation. This version of the force equation, with viscosity included, is called the *Navier-Stokes equation*.

In the rest of this chapter, we look at smooth (time-steady) flows in which viscosity plays the dominant role. This situation is often called *laminar flow*. Faber describes laminar flow in a more restricted sense, implying that "the fluid can be treated as an assembly of laminae of uniform thickness, whose boudaries remain fixed as the flow moves between them". The more general, and more usual, definition, uses laminar flow to mean smooth, non-turbulent flow. In most applications, we note that a real fluid does not slip freely past a boundary (as was the implicit assumption in potential flow theory), but rather sticks: we will generally use *no-slip* boundary conditions.

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[1] What does $\nabla^2\mathbf{a}$ mean, if $\mathbf{a}$ is a vector? For Cartesian it's simple:

$$\nabla^2\mathbf{a} = (\nabla^2 a_x, \nabla^2 a_y, \nabla^2 a_z)$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$$

is the usual Laplacian operator. For cylindrical and spherical coordinates, $\nabla^2\mathbf{a}$ is a more complicated form – check the vector references I'll put up on the web, or your favorite vectors-in-funny-coordinates reference book.

B. Steady Flow: Cartesian Applications

In particular, in this section and the next, we consider steady flows, where gravity is unimportant. Thus, we are solving

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \rho\nu\nabla^2\mathbf{v} \tag{2.3}$$

and the second derivative gives this equation the propertires of a diffusion equation.

1. FLOW BETWEEN PARALLEL PLATES

In this section, we consider flow along $\hat{\mathbf{x}}$, with gradients in the $\hat{\mathbf{y}}$ direction $\mathbf{v} = (v(y), 0, 0)$. Thus, the system (2.3) simplifies still further to

$$\begin{aligned} \frac{\partial p}{\partial x} &= \rho\nu\frac{\partial^2 v}{\partial y^2} \\ \frac{\partial p}{\partial y} &= 0 \end{aligned} \tag{2.4}$$

Note, we have eliminated the interial term utterly, by picking our geometry carefully: so that the gradients are perpendicular to the velocities. The second of these equations says that the pressure can vary only with x. In the first, then, the first term can be only a function of x, while the second can be only a function of y; thus, both must be constant. Thus, dp/dx = constant, (2.4) becomes an ODE, and our general solution for $v(y)$ comes from

$$\rho\nu v(y) = Ay + B + \frac{y^2}{2}\frac{dp}{dx} \tag{2.5}$$

Thus, the transverse velocity profile is a quadratic in y. Specific examples come from various imposed boundary conditions, as illustrated in Figure 2.2.

One particular case is *plane Poiseuille flow*, in which an external pressure gradient drives a flow through two stationary walls. This is what we introduced in Figure 2.1. The velocity profile here is a centered parabola, the flow being fastest in the middle (as you'll see in the homework). The mass flux per unit width is

$$Q = \int_0^b \rho v(y)dy = -\frac{1}{3\nu}b^3\frac{dp}{dx} \tag{2.6}$$

Other variants, shown in Figure 2.2, will appear in the homework.

2. FLOW IN AN OPEN CHANNEL

This comes from Faber. Let the channel have an open surface, and be inclined to horizontal at an angle α. The open surface allows us to assume the pressure is everywhere equal to atmospheric pressure, so that ∇p is zero. Keep the Cartesian coordinates aligned with the flow bed. The basic equation is then

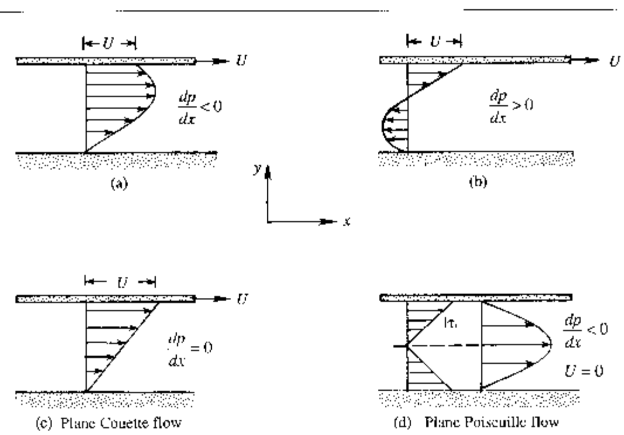

Figure 2.2. Plane-flow solutions for various boundary conditions. From Kundu figure 9.4.

$$\rho\nu\frac{d^2v}{dy^2} = -\rho g\sin\alpha \tag{2.7}$$

The boundary conditions are now $v = 0$ at the lower surface, and $dv/dy = 0$ at the upper (free, open) surface, a distance b away. The latter is due to the fact that shear stresses are continuous at a boundary, and the atmosphere (having very low viscosity) is assumed to carry no shear. This solves to

$$v(y) = \frac{1}{2\nu}g\sin\alpha\left(2by - y^2\right) \tag{2.8}$$

which gives a mass flux,

$$Q = \int_0^b \rho v(y)dy = \frac{1}{3\nu}\rho g b^3\sin\alpha \tag{2.9}$$

3. HELE SHAW FLOW

This describes the means by which the striking two-dimensional images used to illustrate potential flow are made. Inject the fluid through closely spaced parallel plates, with some obstacle in the middle. We want the vertical direction (between the plates; call it $\hat{\mathbf{z}}$) to be governed by viscosity, while the two horizontal directions (say the flow goes along $\hat{\mathbf{z}}$) can be treated as potential flow. Let the plate separation be b, the scale of the central obstacle be L, and the flow be driven by a fixed, externally imposed dp/dx.

First, use our parallel-plate solution to find the discharge rate Q. From this define a characteristic velocity (strictly, the mean over the z-direction):

$$V = \frac{Q}{b} = -\frac{b^2}{12\rho\nu}\frac{dp}{dz} \tag{2.10}$$

(the minus sign just notes that V points opposite to the pressure gradient). But I can write the right hand side as the gradient of a scalar:

$$V = \nabla\left(\frac{b^2}{12\rho\nu}\frac{dp}{dz}\right) \tag{2.11}$$

That is, the horizontal velocity can be treated in terms of a potential, $\phi = -(b^2/12\rho\nu)(d\rho/dz)$. This shows that all the potential flow apparatus of chapter 2 can be used here; and, coversely, that flows where this works are good illustrations for two-dimensional potential flow solutions.

But then: what does it take to justify this? We must be able to ignore the viscous forces in the x and y directions, and to ignore the inertial force in the z direction. That is, we need

$$v_x\frac{\partial v_x}{\partial x} \ll \nu\frac{\partial^2 v_x}{\partial z^2}\ ; \qquad \frac{\partial^2 v_x}{\partial y^2} \ll \frac{\partial^2 v_x}{\partial z^2}$$

But now: the second condition simply requires that $L \gg b$; while the first requires

$$\frac{V^2}{L} \ll \frac{\nu V}{b^2}\ ; \qquad \frac{b^2}{L^2} \ll \frac{1}{\mathrm{Re}} \tag{2.12}$$

Thus, the Reynolds number determines the required dimensions of the apparatus.

C. Steady Flow: Cylindrical Applications

We can repeat the exercise for cylindrical geometry; differences from the planar cases highlight the effects of geometry.

1. PIPE FLOW

Now, consider steady flow in a pipe of radius a. This is *circular Poiseuille flow*. The radial component of the momentum equation again required $dp/dr = 0$, *ie* no transverse pressure gradients. Assuming a very long pipe, so that things only vary with r, the z-component of the momentum equation is

$$\frac{dp}{dz} = \frac{\rho\nu}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) \tag{2.13}$$

As there is only a z-component to the velocity, we have dropped the subscript. By the same argument as used to get (2.5), we argue here that the second term in (2.14) must be constant, giving a flow profile

$$v(r) = \frac{r^2}{4\rho\nu}\frac{dp}{dz} + A\ln r + B \tag{2.14}$$

where A, B are again integration constants, set by the boundary conditions.

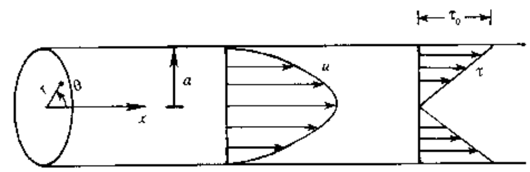

Figure 2.3. The circular Poiseuille flow solution, showing the velocity field and also the stress field (τ; our Σ) From Kundu figure 9.5.

But now, $A = 0$ to keep v finite as $r \to 0$; and a no-slip boundary at $r = a$ gives

$$v(r) = \frac{r^2 - a^2}{4\rho\nu}\frac{dp}{dz} \tag{2.15}$$

that is, another centered parabolic profile. The mass flux here is

$$Q = \int_0^a 2\pi v r dr = -\frac{\pi a^4}{8\nu}\frac{dp}{dz} \tag{2.16}$$

This is called Poiseuille's Law, or the Hagen-Poiseuille law.

2. CIRCULATING FLOW

Another case is flow between two concentric, rotating cylinders, called *circular Couette flow*.

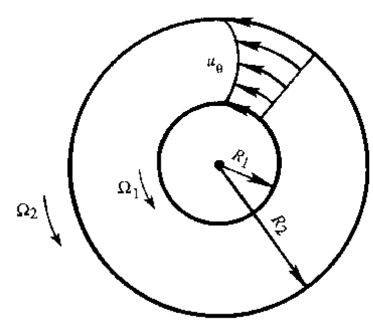

Figure 2.4. The geometry and the velocity field for circular Couette flow. From Kundu figure 9.6.

Referring to (2.41) and (2.42) in Appendix 2, and noting that only $v_\phi \neq 0$, the two components of the equation of motion are

$$\frac{v_\phi^2}{r} = \frac{1}{\rho\nu}\frac{dp}{dr}$$
$$0 = \nu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv_\phi)\right] \tag{2.17}$$

The general solution for the azimuthal velocity is

$$v_\phi(r) = Ar + \frac{B}{r} \tag{2.18}$$

and the pressure gradient, required to offset centrifugal force, is

$$\frac{dp}{dr} = \frac{\rho}{r}\left(Ar + \frac{B}{r}\right)^2 \tag{2.19}$$

Using boundary conditions $v_\phi(R_1) = \Omega_1 R_1$ and $v_\phi(R_2) = \Omega_2 R_2$ gives the specific values for A and B.

This has some interesing limits. One has $\Omega = \Omega_2$ and $R_1 = 0$, that is a rotating cylindrical tank. This has

$$v_\phi = \Omega r \tag{2.20}$$

showing that the fluid inside goes into solid body rotation. Alternatively, if the outer cylinder is at infinity, with $\Omega_2 = 0$, and the inner one has $R = R_1$ and $\Omega = \Omega_1$, then the solution is

$$v_\phi = \frac{\Omega R^2}{r} \tag{2.21}$$

We'll see this again in chapter 4 when we work with the irrotational vortex.

D. Viscous Stresses, Generally

OK, we've avoided this long enough .. we need to write down the more general form of the viscous stresses. This will be a tensor. We start by determining the surface forces on a piece of fluid, due to deformations (arising from velocity gradients) of the adjacent bits of fluid.

1. DO IT PHYSICALLY FIRST

Following Faber here. Consider a planar flow in the $\hat{\mathbf{x}}$ direction, with transverse velocity gradients (along $\hat{\mathbf{y}}$). We expect the shear stress to be linear: $\nu\hat{\mathbf{x}}dv_x/dy$ is the force between adjacent streams in the fluid.[2] (Note that the shear force points along the interface between the two fluid parcels). The coefficient ν is the *viscosity* or *shear viscosity*.

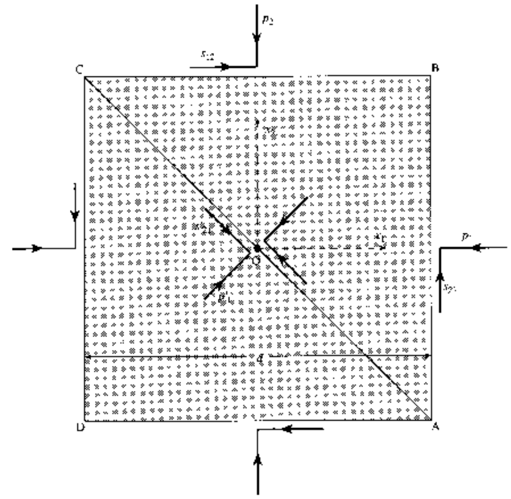

Figure 2.5. The two-dimensional geometry for the stress tensor. p_i are the normal forces on the fluid square, and s_{ij} are the shear forces. From Faber, Figure 1.2

Life is a bit more complicated, however. Consider a two-dimensional Cartesian system, where s_{12} is the force in direction 2, due to adjacent fluid in direction 1; and s_{21} is the force in direction 1, due to adjacent fluid in direction 2. These two forces must be equal: $s_{12} = s_{21}$; if not we would have a net torque on the parcel of fluid. Thus, we must symmetrize our expression for the shear force:

$$s_{12} = s_{21} = \rho\nu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = s_3 \tag{2.22}$$

where the last equality simply defines s_3, to shorten the notation.

There are also normal forces on the parcel: due to the fluid pressure, and also due to the fluid deformation. Faber approaches this by considering a frame rotation, into the $45°$ degree primed frame. From force balance, we find that

$$s_3' = \frac{1}{2}(p_1 - p_2)$$
$$p_1' = \frac{1}{2}(p_1 + p_2 - 2s_3) \tag{2.23}$$

and also, from a Taylor expansion, we find that

$$s_3 = \rho\nu\left(\frac{\partial v_1'}{\partial x_1'} - \frac{\partial v_s'}{\partial x_2'}\right)$$

from which we get

$$p_1 + 2\rho\nu\frac{\partial v_1}{\partial x_1} = p_2 + 2\rho\nu\frac{\partial v_2}{\partial x_2} = p_3 + 2\rho\nu\frac{\partial v_3}{\partial x_3} \tag{2.24}$$

[2] This is a BIG assumption – its traditional, but not obvious to me that it should always hold. Look ahead to the effects of a magnetic field and $\mathbf{j}\times\mathbf{B}$ forces, for a possible counter-example. Luckily, however, we rarely need to worry about viscous stresses in MHD applications ....

But now, taking the mean (isotropic) pressure to be

$$p = \frac{1}{3}(p_1 + p_2 + p_3) \quad (2.25)$$

we end up with an expression for the normal force:

$$p_1 = p - \frac{2}{3}\rho\nu\left(2\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3}\right) \quad (2.26)$$

So, generalizing to three dimensions, we have all 9 components (6 independent) of the surface forces acting on this parcel, which we collect as the tensor $\vec{\vec{\sigma}}$:

$$\sigma_{ij} = -\begin{bmatrix} p_1 & s_{12} & s_{13} \\ s_{21} & p_2 & s_{23} \\ s_{31} & s_{32} & p_3 \end{bmatrix} \quad (2.27)$$

(the minus sign has been added for consistency with the usual treatment).

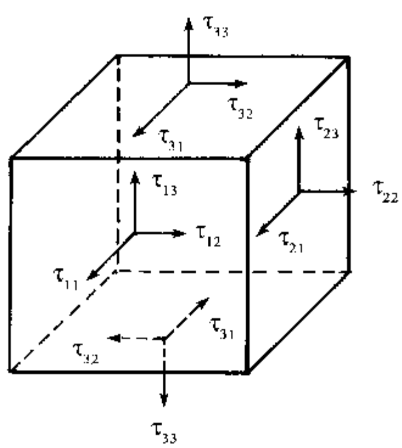

Figure 2.6. The surface forces on a three-dimensional fluid cube. The notation here uses τ for our σ. The normal forces are τ_{ii}, and the shear forces are τ_{ij}. From Kundu figure 2.2.

2. THEN DO IT FORMALLY

We can also approach this more formally (following Thompson). First, we want to write the surface forces as a general stress tensor,

$$\sigma_{ij} = -p\delta_{ij} + \Sigma_{ij} \quad (2.28)$$

Next, write a general velocity gradient in terms of symmetric (D_{ij}) and antisymmetric (Ω_{ij}) parts:

$$\frac{\partial v_i}{\partial x_j} = D_{ij} + \Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_j}\right) \quad (2.29)$$

and we note that the Ω_{ij} term is 1/2 of the usual vorticity, $\Omega = \nabla \times \mathbf{v}$. Also, note that the trace of D_{ij} is

$$D_{mm} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \nabla \cdot \mathbf{v} \quad (2.30)$$

Now, we argue that Σ_{ij} should depend only on the *symmetric* part of the velocity gradient, D_{ij} (otherwise, simple rotation, such as solid body, would lead to a spurious stress term). Thus, requiring linearity, we have in general

$$\Sigma_{ij} = \alpha_{ijmn} D_{mn} \quad (2.31)$$

which involves no fewer than 81 separate α's. However, we can make arguments about symmetry and isotropy to reduce this to the most general form useful for us:

$$\Sigma_{ij} = 2\rho\nu\left(D_{ij} - \frac{1}{3}\delta_{ij}D_{mm}\right) + \rho\nu_b\delta_{ij}D_{mm} \quad (2.32)$$

That is, we have only two α's – ν, the *shear viscosity*, usually just called the viscosity, and ν_b, the *bulk viscosity*. Thus, the general stress tensor is

$$\sigma_{ij} = -p\delta_{ij} + 2\rho\nu\left(D_{ij} - \frac{1}{3}\delta_{ij}D_{mm}\right) + \rho\nu_b\delta_{ij}D_{mm} \quad (2.33)$$

From this, we have the full expression for the vector surface force per area:

$$\mathbf{T} = \hat{\mathbf{e}_k}\sigma_{ik} \quad (2.34)$$

if $\hat{\mathbf{e}_k}$ is the unit vector in the kth direction.

In many applications the bulk viscosity is ignored. This is called the *Stokes hypothesis*; Thompson discusses it nicely. This hypothesis is physically true (at the kinetic level) only for simple systems – dilute, monatomic gases. For other fluids, $\nu_b \gtrsim \nu$ is possible. However, in many applications the flows are incompressible, and (from 2.30 or 2.33) the effects of the bulk viscosity term can be ignored.

E. The Navier-Stokes Equation (in Cartesian)

Now, we want to add viscous stresses to the force equation. Remember that we're still working in Cartesian. We need to generalize the ∇p term in the Euler equation (*e.g.*, 1.10) to the full stress tensor. With the stress term added, the force equation is called the *Navier-Stokes equation*. Recall, $\mathbf{g} = -\nabla\Phi$ is the body force (derivable from a potential). In Cartesian, in index notation, we have the form

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \rho g_i + \frac{\partial \sigma_{ki}}{\partial x_k} \tag{2.35}$$

and in vector notation, we have the form

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \rho\mathbf{g} + \rho\nu\nabla^2\mathbf{v} + \left(\rho\nu_b + \frac{1}{3}\rho\nu\right)\nabla(\nabla\cdot\mathbf{v}) \tag{2.36}$$

If Stokes' hypothesis holds, this simplifies a bit to

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \rho\mathbf{g} + \rho\nu\nabla^2\mathbf{v} + \frac{1}{3}\rho\nu\nabla(\nabla\cdot\mathbf{v}) \tag{2.37}$$

which is still general (allows variation of ρ). If we now assume the fluid is incompressible, $\nabla\cdot\mathbf{v} = 0$, (2.37) becomes

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = -\rho\nabla\Phi + \rho\nu\nabla^2\mathbf{v} \tag{2.38}$$

which is the most commonly used form of this important equation, and recovers our "guess" in (2.2).

References

I've mostly followed Kundu and Faber here, as well as Thompson (whose discussion of viscosity and stress tensors I like). For the non-Cartesian forms of tne NS equation, Pozrikidis is one good reference.

F. Appendix: Navier-Stokes in other coordinates

It is also worth storing the stress tensor and the N-S equation in curvilinear coordinates (taken from Pozrikidis). Here, we are using only the shear viscosity, and dropping the bulk viscosity term.

1. Cylindrical coordinates

The 6 independent components of the stress tensor are

$$\begin{aligned}
\sigma_{rr} &= -p + 2\rho\nu\frac{\partial v_r}{\partial r} & \sigma_{r\phi} &= \rho\nu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{2}{r}\frac{\partial v_r}{\partial \phi}\right] \\
\sigma_{rz} &= \rho\nu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) & \sigma_{\phi\phi} &= -p + \rho\nu\left(\frac{2}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{2}{r}v_r\right) \\
\sigma_{\phi z} &= \rho\nu\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right) & \sigma_{zz} &= -p + 2\rho\nu\frac{\partial v_z}{\partial z}
\end{aligned} \tag{2.39}$$

and the 3 componts of the NS equation are:

$$\frac{\partial v_r}{\partial t} + v_z \frac{\partial v_r}{\partial z} + v_r \frac{\partial v_r}{dr} + \frac{v_\phi}{r}\frac{\partial v_\phi}{d\phi} - \frac{v_\phi^2}{r} = g_r + \frac{1}{\rho}\left(\frac{\partial \sigma_{zr}}{\partial z} + \frac{1}{r}\frac{\partial(r\sigma_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\phi}}{\partial \phi} - \frac{\sigma_{\phi\phi}}{r}\right) \tag{2.40}$$

$$\frac{\partial v_\phi}{\partial t} + v_z \frac{\partial v_\phi}{\partial z} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} = g_\phi + \frac{1}{\rho}\left(\frac{\partial \sigma_{z\phi}}{\partial z} + \frac{1}{r^2}\frac{\partial(r^2\sigma_{r\phi})}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\phi\phi}}{\partial \phi}\right) \tag{2.41}$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi} = g_z + \frac{1}{\rho}\left(\frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\frac{\partial(r\sigma_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\phi z}}{\partial \phi}\right) \tag{2.42}$$

Alternatively, we can write the NS equations out explicitly for the case of constant viscosity:

$$\begin{aligned}&\frac{\partial v_r}{\partial t} + v_z \frac{\partial v_r}{\partial z} + v_r \frac{\partial v_r}{dr} + \frac{v_\phi}{r}\frac{\partial v_\phi}{d\phi} - \frac{v_\phi^2}{r} \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left[\frac{\partial^2 v_r}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}\right]\end{aligned} \tag{2.43}$$

$$\begin{aligned}&\frac{\partial v_\phi}{\partial t} + v_z \frac{\partial v_\phi}{\partial z} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} \\ &\quad = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + g_\phi + \nu\left[\frac{\partial^2 v_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\phi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r}\frac{\partial v_r}{\partial \phi}\right]\end{aligned} \tag{2.44}$$

$$\begin{aligned}&\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi} \\ &\quad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\left[\frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2}\right]\end{aligned} \tag{2.45}$$

2. Spherical Polar Coordinates

The 6 independent components of the stress tensor are

$$\begin{aligned}\sigma_{rr} &= -p + 2\rho\nu\frac{\partial v_r}{\partial r} & \sigma_{r\theta} &= \rho\nu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \\ \sigma_{r\phi} &= \rho\nu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] & \sigma_{\phi\phi} &= -p + \frac{\rho\nu}{r\sin\theta}\left(2\frac{\partial v_\phi}{\partial \phi} + v_r\sin\theta + v_\theta\cos\theta\right) \\ \sigma_{\theta\phi} &= \rho\nu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] & \sigma_{\theta\theta} &= -p + \rho\nu\left(\frac{2}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{2v_\theta}{r}\right)\end{aligned} \tag{2.46}$$

and the 3 componts of the NS equation are:

$$\begin{aligned}&\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \\ &\quad + g_r + \frac{1}{\rho}\left(\frac{1}{r^2}\frac{\partial(r^2\sigma_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r}\right)\end{aligned} \tag{2.47}$$

$$\begin{aligned}&\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - v_\phi^2\frac{\cot\theta}{r} \\ &\quad + g_\theta + \frac{1}{\rho}\left(\frac{1}{r^2}\frac{\partial(r^2\sigma_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{r\theta}}{r} - \frac{\sigma_{\phi\phi}}{r}\cot\theta\right)\end{aligned} \tag{2.48}$$

$$\begin{aligned}\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi}{r}\left(v_r+v_\theta\cot\theta\right)\\ =g_\phi+\frac{1}{\rho}\left(\frac{1}{r^2}\frac{\partial(r^2\sigma_{r\phi})}{\partial r}+\frac{\sigma_{r\phi}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta\phi}}{\partial\theta}+\frac{2\cos\theta}{r}\sigma_{\theta\phi}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\right)\end{aligned} \tag{2.49}$$

Once again, we can write these out explicitly:

$$\begin{aligned}\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\\ =-\frac{1}{\rho}\frac{\partial p}{\partial r}+g_r+\nu\left[\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\left(\frac{\partial v_\theta}{\partial\theta}+v_\theta\cot\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right]\end{aligned} \tag{2.50}$$

$$\begin{aligned}\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-v_\phi^2\frac{\cot\theta}{r}\\ =-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+g_\theta+\nu\left[\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right]\end{aligned} \tag{2.51}$$

$$\begin{aligned}\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi}{r}\left(v_r+v_\theta\cot\theta\right)\\ =-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi}+g_\phi+\nu\left[\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right]\end{aligned} \tag{2.52}$$

In this set, for f some scalar function,

$$\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

and this is the end of this chapter.