20. IDEAL MHD INSTABILITIES

We now turn to MHD instabilities. They are critical to fusion plasmas, where one wants to confine the plasma and keep it stable long enough for fusion to take place. Thus, an extensive literature has developed on ideal (non-resistive) instabilities of MHD equilibria.

A. Overview; Energy Methods

The history of this field comes, of course, from the need to keep plasmas around in the lab long enough for them to fuse (Bateman has a good discussion of this period; Figure 20.1 illustrates the basic pictures). In early experiments, a current was driven through a column of ionized gas, in an effort to heat and confine it. These early experiments found that the plasma column would spontaneously pinch itself off in a process that came to be known as the $m = 0$ *sausage instability*. This instability can easily be stabilized by adding an axial magnetic field to the plasma column (requiring an azimuthal current). However, this configuration introduced a new instability, in which the plasma column twists into a helical or corkscrew shape; this is the $m = 1$ *kink instability*. Higher m modes may be considered, of course; $m = 2$ is shown in the figure.

1. ENERGY METHODS

MHD instabilities often involve more complex geometries than the fluid instabilities we have considered. Thus, a full modal analysis which describes relevant (macroscopic) perturbations is often too ugly to consider. A different approach is available here: the *energy method*. This analysis allows us to determine if a system is stable or unstable; it does not, however, determine growth rates or, necessarily, details of the nature of the unstable perturbation.

To illustrate this, consider a simple 1D system, with a potential energy curve $W(x)$, connected to a force $F = -dW/dx$. Equilibrium will be an extremum of $W(x)$, say x_o; but as in Figure 20.2, it can be either a stable (minimum of W) or unstable (maximum of W) extremum.

We can test this by considering a small displacement, say from x_o to x. With this, the change in the potential energy is

$$\delta W = W(x) - W(x_o) \simeq \frac{1}{2}(x - x_o)^2 \left(\frac{d^2W}{dx^2}\right)_o \tag{20.1}$$

In this last step we have used $dW/dx = 0$ at the extremum. Thus, the sign of δW determines the stability. In particular, if we consider some displacement δx

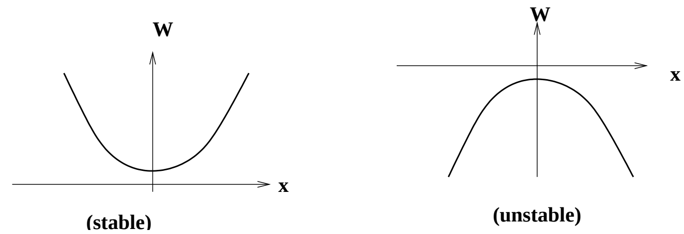

Figure 20.1. Stable and unstable potential energy curves, in a simple one-dimensional illustration. Let the energy curve $W(x)$ have an extremum at some value of x: $dW/dx = 0$. If the energy W is a minimum here, the system is stable; a small perturbation will return to this point. If, however, W is a maximum there, the system is unstable; a small perturbation will evolve away from this point.

and find $\delta W < 0$, we know that we have an unstable system. Given a slight push, such a system will move away from its initial equilibrium state, towards a state of lower energy; thus it is unstable. Conversely, if we can slow $\delta W > 0$ for *all* possible perturbations, we know the system is stable.

2. THE DETAILS

This is the idea that we want to generalize to a 3D plasma. I'm following Freiberg's presentation here. A general static plasma configuration has potential energy

$$W = \int \left(\frac{B^2}{8\pi} + \rho e\right) dV \tag{20.2}$$

The first term is the magnetic energy, the second the internal energy, and the integral is taken over the plasma volume. Now, let each piece of the plasma be displaced by some local $\delta\mathbf{x}$. The fluid quantities are then disturbed as

$$\begin{aligned}
\mathbf{v}_1 &= \frac{d\delta\mathbf{x}}{dt} \\
\mathbf{B}_1 &= \nabla \times (\delta\mathbf{x} \times \mathbf{B}_o) \\
\rho_1 &= -\nabla \cdot (\rho_o \delta\mathbf{x}) \\
p_1 &= \frac{\gamma p_o}{\rho_o}\rho_1 - p_o(\delta\mathbf{x} \cdot \nabla) \ln \frac{p_o}{\rho_o^\gamma}
\end{aligned} \tag{20.3}$$

(The first of these should be obvious; the last 3 of these come from the D/Dt expressions, integrated once with resepect to time).

Now, the (linearized) force on that piece of the plasma due to the perturbation is

$$\mathbf{F}(\delta\mathbf{x}) = -\nabla p_1 + \frac{\mathbf{j_1}}{\mathbf{c}} \times \mathbf{B}_o + \frac{\mathbf{j}_o}{c} \times \mathbf{B}_1 \tag{20.4}$$

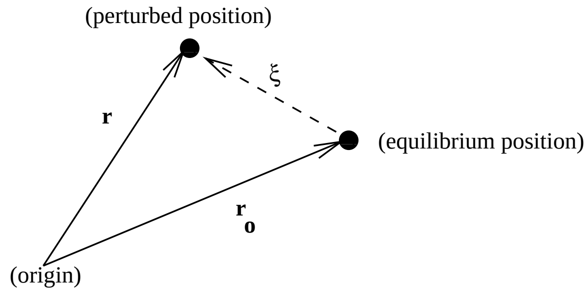

Figure 20.2. A small displacement of a plasma element by $\vec{\xi}$ from its equilibrium position ($\mathbf{r}_o$) to a new position $\mathbf{r}$ . . . relative to the origin O. Our $\delta\mathbf{x}$ is $\vec{\xi}$ in this figure. Following Priest Figure 17.2.

and the change in the potential energy, due to this perturbation, is

$$\delta W = -\frac{1}{2}\int \delta\mathbf{x}\cdot\mathbf{F}dV \qquad (20.5)$$

where the explicit form (20.4) is used in the integrand (the 1/2 is there to take a rough mean over the displacement from 0 to $\mathbf{x}$.) Expanding this out, and writing $\mathbf{B}_1 = \nabla\times(\delta\mathbf{x}\times\mathbf{B}_o)$, one explicit form for the perturbation of the "fluid+field" energy is

$$\delta W = \frac{1}{2}\int\left[\frac{B_1^2}{4\pi} - \delta\mathbf{x}_\perp\cdot\frac{\mathbf{j}}{c}\times\mathbf{B}_1 + \gamma p|\nabla\cdot\delta\mathbf{x}|^2 + (\delta x_\perp\cdot\nabla p)\nabla\cdot\delta\mathbf{x}_\perp\right]dV \qquad (20.6)$$

In addition, many applications require the addition of a "boundary" or "surface" term:

$$\delta W_S = \frac{1}{2}\int\left[\frac{\mathbf{B}\cdot\mathbf{B}_1}{4\pi} - \gamma p\nabla\cdot\delta\mathbf{x} - \delta\mathbf{x}_\perp\cdot\nabla p\right]\delta\mathbf{x}\cdot\hat{\mathbf{n}}dS \qquad (20.7)$$

An alternative version of this which explicitly uses the field line (inverse) curvature, $\vec{\kappa} = \mathbf{b}\cdot\nabla\mathbf{b}$ if $\mathbf{b}$ is the unit vector along $\mathbf{B}$, is

$$\delta W = \frac{1}{2}\int\left[\frac{B_{1\perp}^2}{4\pi} + \frac{B^2}{4\pi}|\nabla\cdot\delta\mathbf{x}_\perp + 2\delta\mathbf{x}_\perp\cdot\vec{\kappa}|^2 + \gamma p|\nabla\cdot\delta\mathbf{x}|^2 - 2(\delta\mathbf{x}_\perp\cdot\nabla p)(\vec{\kappa}\cdot\delta\mathbf{x}_\perp) - \frac{j_\parallel}{c}(\mathbf{x}_\perp\times\mathbf{b})\cdot\mathbf{B}_1\right]dV \qquad (20.8)$$

This last form is the so-called "intuitive form"[1]; each term allows a simple physical interpretation. The $\mathbf{B}_{1\perp}$ term represents the energy needed to bend field lines. The second term is the energy needed to compress the magnetic field. The γp term, the third term, is the energy needed to compress the plasma. Each of these contributions is stabilizing. The last two terms can be positive or negative, and thus can drive instabilities. The first of these depends on $\nabla p \sim \mathbf{j}_\perp\times\mathbf{B}$, while the second depends on the parallel current $j_\parallel$. Thus, a vacuum field is stable (but not very interesting), while either perpendicular or parallel currents are potential sources of instability.

Thus: the energy method is applied starting with some equilibrium configuration (as in Chapter 17, say), and asking whether *some* perturbation $\delta\mathbf{x}$ can be found for which $\delta W < 0$. If this is the case, then we know our initial configuration is unstable. (The converse isn't necessarily useful: if we try some perturbation and find $\delta W > 0$, this doesn't prove stability unless *all* perturbations have been considered).

B. Apply: Pinch Instabilities

1. THETA PINCH

We first look at stability of the θ pinch: which is simply described (*cf.* §17.3) by

$$p(r) + \frac{B_z^2}{8\pi} = \frac{B_o^2}{8\pi} \qquad (20.9)$$

[1] oh yeah?

if B_o is the confining field. We recall that this needs an outer wall, to hold everything together; it is not a self-confined system. We will find, however, that it is stable to the type of MHD perturbations we're considering here.

Consider displacements

$$\delta\mathbf{x}(\mathbf{r}) = \delta\mathbf{x}(r)e^{i(m\phi+kz)} \tag{20.10}$$

and write $\delta\mathbf{x} = (\xi_r, \xi_\phi, \xi_z)$. First, note that the incompressibility condition $\nabla\cdot\mathbf{v} = 0$ gives

$$-ikr\xi_z - im\xi_\phi = \frac{d}{dr}(r\xi_r) \tag{20.11}$$

The terms that go into the integrand for δW are

$$\begin{aligned}\mathbf{B}_{1\perp} &= ikB_z\mathbf{k}_\perp = ikB_z(\xi_r, \xi_\phi, 0)\\ \nabla\cdot\delta\mathbf{x}_\perp &= \frac{1}{r}\frac{d}{dr}(r\xi_r) + \frac{im}{r}\xi_\phi\\ \vec{\kappa} &= 0\\ j_\parallel &= 0\end{aligned} \tag{20.12}$$

Putting these into (20.7), taking a as the pinch radius and L as its length, we have

$$\delta W = \frac{\pi L}{4\pi}\int_0^a \delta W(r)dr \tag{20.13}$$

where the integrand can be written (with $k_o^2 = k^2 + m^2/r^2$)

$$\begin{aligned}\frac{\delta W(r)}{B_z^2} &= \left(k_o\xi_\phi - \frac{im}{k_o r^2}\frac{d}{dr}(r\xi_r)\right)^2\\ &\quad + \frac{k^2}{k_o^2 r^2}\left(\frac{d}{dr}(r\xi_r)\right)^2 + k^2\xi_r^2\end{aligned} \tag{20.14}$$

Now, the first term is the only one containing ξ_ϕ: the perturbed δW can therefore be minimized by choosing

$$\xi_\phi = \frac{im}{m^2+k^2r^2}\frac{d}{dr}(r\xi_r) \tag{20.15}$$

But then, with this choice, the integrand in (20.13) for δW is positive definite for any choice of ξ_r. Thus, the *minimum of* δW is positive for $k > 0$, and approaches zero as $k \to 0$: this system is stable for finite wavelengths and approaches marginal stability for very long wavelengths.

What is the underlying physics? This equilibrium has no parallel currents, so current-driven modes (such as the pinch) can't be excited. In addition, as the field lines are straight, pressure-driven modes (such as the kink) can be excited. Any perturbation to this equilibrium either bends or compresses the field lines, and both are stabilizing influences.

2. Z PINCH

Now, let's do an unstable one. The equilibrium here is

$$\frac{dp}{dr} + \frac{B_\phi}{4\pi r}\frac{d}{dr}(rB_\phi) = 0 \tag{20.16}$$

This is self-confined; all pressures drop at large radii. This system is particularly interesting if the source current is carried by the plasma itself. However, we will see that it is severly unstable to large-scale perturbations which can disrupt the equilibrium.

We again consider a perturbation (20.9); but now the $m = 0$ and $m \neq 0$ modes must be considered separately. We will work through $m \neq 0$ here. The expressions analogous to (20.11) are

$$\begin{aligned}\mathbf{B}_{1\perp} &= \frac{imB_\phi}{r}(\xi_r, 0, \xi_z)\\ \vec{\kappa} &= -(1/r, 0, 0)\\ \nabla\cdot\delta\mathbf{x}_\perp + 2\delta\mathbf{x}_\perp\cdot\vec{\kappa} &= r\frac{d}{dr}\left(\frac{\xi_r}{r}\right) + ik\xi_z\\ j_\parallel &= 0\end{aligned} \tag{20.17}$$

Again evaluating $\delta W(r)$ (to go in 20.12), one finds

$$\begin{aligned}\delta W(r) &= \frac{m^2B_\phi^2}{r^2}\left(\xi_r^2+\xi_z^2\right)\\ &\quad + B_\phi^2\left[r\frac{d}{dr}\left(\frac{\xi_r}{r}\right) + ikr\xi_z\right]^2 + \frac{8\pi}{r}\frac{dr}{dr}\xi_r^2\end{aligned} \tag{20.18}$$

Now, this expression can be rearranged so that ξ_z appears only in quadrature. Thus, as with ξ_ϕ above, the energy perturbation is minimized here if

$$\xi_z = \frac{ikr^2}{m^2+k^2r^2}\frac{d}{dr}\left(\frac{\xi_r}{r}\right) \tag{20.19}$$

When this is chosen, the energy perturbation can be written

$$\begin{aligned}\delta W(r) &= \left(8\pi r\frac{dp}{dr} + m^2B_\phi^2\right)\frac{\xi_r^2}{r^2}\\ &\quad + \frac{m^2r^2B_\phi^2}{m^2+k^2r^2}\left(\frac{d}{dr}\left(\frac{\xi_r}{r}\right)\right)^2\end{aligned} \tag{20.20}$$

Still hunting for a minimum of δW, we note that $k \to \infty$ (long wavelengths) kills the last term. Thus, our final expression to be analyzed for the energy perturbation is

$$\delta W = \frac{\pi L}{4\pi}\int_0^a \left(8\pi r\frac{dp}{dr} + m^2B_\phi^2\right)\frac{\xi_r^2}{r^2}r dr \tag{20.21}$$

And now: the magnitude and sign of dp/dr control the result. In particular, if

$$2r\frac{dp}{dr} + \frac{m^2 B_\phi^2}{4\pi} < 0 \tag{20.22}$$

anywhere, then one can find a $\delta\mathbf{x}$ function localized in this region which will make the full integral negative. Thus, this equilibrium *is unstable against* $m \neq 0$ *perturbations*. This is the classic *kink instabilitey*.

If we use the starting equilibrium relation, (20.15), the stability condition (20.21) can be rewritten in two ways:

$$\begin{aligned} \frac{1}{B_\phi^2}\frac{d}{dr}\left(rB_\phi^2\right) &< m^2 - 1 \\ \frac{2r^2}{B_\phi^2}\frac{d}{dr}\left(\frac{B_\phi}{r}\right) &< m^2 - 4 \end{aligned} \tag{20.23}$$

Now, for typical Z pinches, B_ϕ/r is a decreasing function of r. For such profiles, (20.22) predicts stability for $m \geq 2$. Also, at large radii $B_\phi \sim 1/r$ corresponding to a vacuum field; but near the origin, in the current-carrying region, $B_\phi \sim r$. Thus, a Z pinch is always instable to the $m = 1$ mode. Figure 20.7 shows the physical picture of this instability. Physically, the kink instability may be attributed to a localized twist in the column, pushing the field lines together on the inside edge of the kink, and pulling them apart on the outside edge. This sets up an internal magnetic pressure gradient which acts to enhance the kink.

A similar analysis can be done for $m = 0$ perturbations; it turns out that one can't assume incompressibility in this case, so it must be treated separately. One finds an analogous result: the system is unstable if

$$-\frac{r}{p}\frac{dp}{dr} > \frac{2\gamma B_\phi^2/4\pi}{\gamma p + B_\phi^2/4\pi} \tag{20.24}$$

This is called the *pinch instability* or the *sausage instability*. This physical picture for this instability is given in Figure 20.5. Physically, we can attribute the pinch instability to a local constriction in the plasma column causing a local increase in $j_\|$, and thus of the value of B_ϕ there. This provides an extra inwards tension, which tends to enhance the constriction.

We can stabilize a Z pinch by adding an axial magnetic field, B_z. For the pinch mode, the axial field provides a pressure which fights back against the constricting B_{phi} field. For the kink mode, the axial field provides a tension which fights back against the tendency of the azimuthal field to kink. General stability requires a particular mix of axial and azimuthal fields. In particular, it can be shown that the axial magnetic field must be made strong enough, and the plasma column fat enough, that no part of the field between the plasma and the wall closes on itself over the length of the plasma column. This is the *Kruskal-Shafranov* stability criterion, which can be written

$$rB_z(r) > LB_\phi(r) \tag{20.25}$$

if L is the length of the plasma column. Thus, this imposes a severe limit on the current than can be driven through the plasma.

References

I've mostly followed a mixture of Priest & Bateman, here.