21. RESISTIVE MHD INSTABILITIES

In the previous section we considered ideal MHD instabilities – those for which resistivity is not important. Another very important type of MHD behavior involves the effects of resistivity. Resistivity allows plasma to move across magnetic field lines. This violates magnetic flux conservation (what we called "frozen in field" or "flux freezing"). Unlike the ideal case, resistive effects allow changes in the topology of the field lines. The most important application of this is in field line reconnection, which we treat in the next chapter.

In this chapter we consider resistive instabilities, in which spontaneous field line readjustment occurs. Resistive instabilities are most relevant in a current sheet. That is, a thin region, generally formed at the interface of two different plasmas, whre $\mathbf{B}$ varies rapidly. Several such instabilities are known; here we consider the *tearing mode*, which is fundamental to magnetic reconnection.

A. Tearing Mode: the Physics

The simplest illustration of this effect is a planar, two-dimensional problem. Think about a current sheet (also called a neutral sheet); this is a region over which the field direction switches rapidly. To pick a geometry, let $\mathbf{B}(x = 0)$ to lie totally in the yz plane. We'll call this symmetry plane the "tearing plane" or "tearing layer". There are two important pieces of physics here. One is that parallel currents attract; a free current sheet with no hinderance would tend to bunch together. The second is that plasma has inertia, and can only move across field lines on a slow, diffusion timescale.

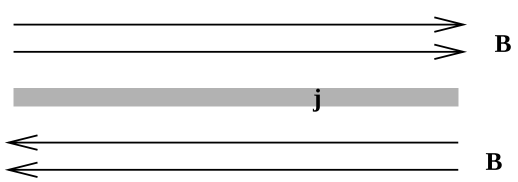

Figure 21.1. Schematic drawing of a neutral sheet configuration which is subject to the tearing mode. Note that there must be a current, **j**, in the region where the field lines switch direction.

Now, imagine a small plasma displacement, perpendicular to the current sheet, and with some fixed wavenumber k. If the plasma is unimportant (has very low energy density compared to the field), then the current will bunch as expected. More important however, is the case when the plasma pressure/inertia matters. First think about the ideal case. The fluid must move with the plasma; so the perturbation will only "wiggle" the field lines – as shown in the top panel of Figure 21.2. Now, consider a finite resistivity. This will allow the field to diffuse through the fluid (or, vice versa, will allow the fluid to move across the field lines). Under the right conditions (mainly long wavelength disturbances; see below), the new, reconnected state has lower energy than the initial one. Thus the system is unstable.

In this geometry, the unstable tearing mode causes magnetic surfaces close to the tearing layer disrupt and reconnect, forming a chain of filaments, as illustrated in Figure 21.2. These filaments are also called "magnetic islands", but note this really only describes a two-dimensional slice of the system.

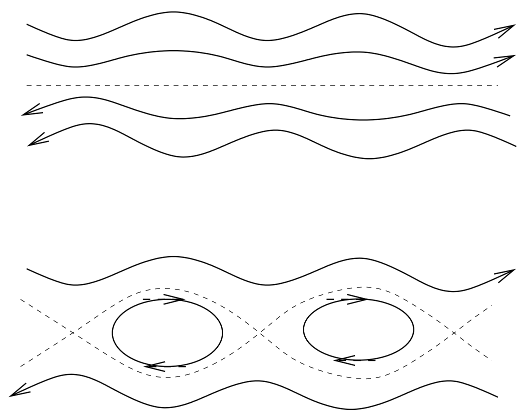

Figure 21.2. Effect of a perpendicular plasma displacement in a neutral sheet, leading to field compression for the ideal case ($\eta = 0$; top figure) and to field disruption and island formation for the resistive case ($\eta \neq 0$; lower figure). Following Biskamp Figure 4.4.

Thus, the main features of tearing modes are: (1) they are driven by field energy associated with shear; (2) they alter B field topology and quickly relax the shear, by field line reconnection and magnetic island formation; (3) the relaxation of the magnetic shear occurs much more quickly than the pure resistive diffusion time would predict.

We will be interested in stability and growth rates. If we have a current layer of thickness a, recall the two characteristic timescales. The Alfven time is

$$\tau_A = a/v_A \ , \tag{21.1}$$

where $v_A = B/(4\pi\rho)^{1/2}$ is the Alfven speed; and the resistive/diffusion time is

$$\tau_D = a^2/\eta \tag{21.2}$$

(or $4\pi a^2/\eta$, depending on author).. In most plasmas of interest, $\tau_A \ll \tau_D$ (recall the high conductivity, and thus small η, for low density plasmas). It turns out the tearing mode goes on an intermediate timescale: $\tau_A \ll \tau_{tear} \ll \tau_D$.

B. Tearing Mode: the Math

I mostly follow Biskamp in these notes; note however that the original derivation, given by Furth, Kileen & Rosenbluth (1963) is quite different on the surface, and is often quoted. Both sources do agree on the answer, however.

We start in the usual place, with the equations of motion (10.2) and induction (10.4). This instability is well described in the incompressible limit; to shorten the notation we set $\rho = 1$, If we linearize the basic equations, and ignore both pressure effects and fluid viscosity, we get

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{v}_1 &= (\nabla\times\mathbf{B}_1)\times\mathbf{B}_o + (\nabla\times\mathbf{B}_o)\times\mathbf{B}_1 \\ \frac{\partial}{\partial t}\mathbf{B}_1 &= \nabla\times(\mathbf{v}_1\times\mathbf{B}_o) + \eta\nabla^2\mathbf{B}_1 \end{aligned} \tag{21.3}$$

Assuming incompressibility allows us to introduce a velocity stream function, ϕ. Working in two dimensions allows us to use a magnetic flux function, ψ. In this coordinate system, we thus have

$$\mathbf{v} = \hat{\mathbf{z}}\times\nabla\phi\,; \qquad \mathbf{B} = \hat{\mathbf{z}}\times\nabla\psi \tag{21.4}$$

(Strictly speaking, these are the perpendicular velocity and field components; in the details of the analysis we're ignoring variations in the y direction). Resistive instabilities are driven mainly by the parallel current: $\mathbf{j} \simeq j_{\|} \simeq j_z$. Therefore we also neglect $j_\perp$, which means $B_z \simeq$ constant. Also, as we're interested in stability and the growth rate, we write $\partial f/\partial t = \gamma f$ (so that γ real and positive corresponds to instability) and make solving for γ the primary goal of our analysis. With this, the equations (21.3) become

$$\begin{aligned} \gamma\nabla_\perp^2\phi_1 &= \mathbf{B}\cdot\nabla j_1 + \mathbf{B}_1\cdot\nabla j \\ \gamma\psi_1 &= \mathbf{B}\cdot\nabla\phi_1 + \eta\nabla_\perp^2\psi_1 \end{aligned} \tag{21.5}$$

Our plan now is (1) pick a geometry for the unperturbed field; (2) solve these equations for the spatial behavior of ψ_1 and ϕ_1, and (3) from these results determine γ. Remember that dissipation is a second-order derivative term; it will be important only in small regions where things change rapidly. In this problem that means close to the current sheet. The traditional way to solve the system (21.5) is in two parts. Close to the current sheet, we keep all the physics, but use simplified geometry to get an answer. We call the width of this inner, resistive region δ. Far from the current sheet, we use simplified physics (that is, we ignore the resistivity). We then require the two solutions to match where they overlap (formally this is an asymptotic analysis), and this matching allows us to solve for γ and δ. One useful quantity in this analysis is the jump in the variation scale of ψ_1 at the tearing layer. Its standard, if ugly, notation is

$$\Delta' = \lim_{\delta\to 0}\frac{1}{\psi_1}\left[\psi_1'(\delta) - \psi_1'(-\delta)\right] \tag{21.6}$$

Here and in the rest of this section, primes on ψ_1 and ϕ_1 mean derivatives with respect to x; the prime on Δ' is standard notation but does not mean it is a derivative (mutter, mutter . . .). We will consider cases where ψ_1 varies only slowly with x near the tearing plane, so that $\psi_1. \ll \psi_1/\delta$, but where the second derivative $\psi_1'' \simeq \psi)_1\Delta'/\delta$, has a jump at $x = 0$.

To do the actual solution, we work first in the inner, resistive region (where the dissipation terms are most important, so we can ignore the inductive terms). We then work in the outer, nearly-ideal region (where we can ignore the dissipation terms). Finally, we require that the two solutions match well, which gives us our final answer for the growth rate (or timescale).

• **Inner, Resistive Region** Now: we pick a wavenumber for the perturbation, $\psi_1(x,y) = \psi_1(x)e^{iky}$, and we assume the unperturbed ψ is constant across this region. Keeping the resistive terms, the system (21.5) becomes

$$\begin{aligned} \gamma\phi_1'' &= ixkB'\psi_1'' - ikj'\psi_1 \\ \gamma\psi_1 &= ixkB'\phi_1 + \eta\psi'' \end{aligned} \tag{21.7}$$

It turns out that j' term does not affect the resulst much, and it is usually ignored in this analysis. Doing that, we solve the system by (1) choosing parity: pick $\psi_1(x)$ even and $\phi_1(x)$ odd; (2) approximate derivatives: $\phi_1'' \simeq -\phi_1/\delta^2$, $\psi_1'' \simeq -\Delta'\phi_1/\delta$. Putting these in and doing some algebra gives us an intermediate solution for the inner region:

$$\begin{aligned} &\gamma \simeq \eta^{3/5}(\Delta')^{4/5}(kB')^{2/5} \\ &\delta \simeq \delta\Delta'/\gamma \simeq \eta^{2/5}(\Delta')^{1/5}(kB')^{-2/5} \end{aligned} \tag{21.8}$$

Biskamp's comments here: this argument assumed $\Delta' > 0$. A full stability analysis (*cf.* FKR 1963) finds that instability requires $\Delta' > 0$, and in fact shows that $-\Delta'$ is the perturbation energy, δW (so that $\delta W < 0$ gives instability). Thus our choice of $\Delta' > 0$ here is OK.

•**Outer, Ideal Region**. In this region, we ignore second derivatives, so the basic equation becomes

$$\mathbf{B}\cdot\nabla j_1 + \mathbf{B}_1\cdot\nabla j \simeq 0 \tag{21.9}$$

Putting in the geometry of Figure 21.1 explicitly, this becomes

$$\psi_1'' - \left(k^2 + \frac{j'(x)}{B(x)}\right) = 0 \qquad (21.10)$$

Now we must pick a form for $B(x)$. The usual choice for this problem is $B(x) = \tanh(x/a)$, which introduces a as the width of the current layer. We pick boundary conditions $\psi_1 \to 0$ as $|x| \to \infty$. The solution of (21.10) is now analytic:

$$\psi_1(x) = e^{-k|x|}\left(1 + \frac{1}{ka}\tanh\left|\frac{x}{a}\right|\right)$$
$$\Delta' = \frac{2}{a}\left(\frac{1}{ka} - ka\right) \qquad (21.11)$$

Recalling that a full analysis shows instability only when $\Delta' > 0$, we see from this that only large scales are unstable: $ka < 1$, or $\lambda > 2\pi/a$

• **Put These Together**. All this collects to give our final result, the growth rate and the physical scale of the inner region. We can find these by putting the solution (21.11) back into the intermediate solution (21.8). Doing the algebra, and working in terms of the two time scales defined at the start, we find for $ka \ll 1$,

$$\gamma \simeq \frac{1}{(ka)^{2/5}} \frac{1}{\tau_D^{3/5}\tau_A^{2/5}} \qquad (21.12)$$

and for the layer thickness,

$$\delta \sim \frac{a}{(ka)^{3/5}}\left(\frac{\tau_A}{\tau_D}\right)^{2/5} \qquad (21.13)$$

Thus, we find that such a current sheet is always unstable to tearing mode on long scales (low k), and that the growth time ($\tau_{tear} \simeq 1/\gamma$) is, indeed, inbetween the Alfven (dynamic) and resistive (diffusion) timescales.

C. Tearing Mode: the Consequences

The evolution of the tearing mode in a simple geometry is illustrated by the figures on the next page, from an early numerical simulation. Magnetic island formation (that is flux ropes, thinking of the third dimension) does take place as expected, and currents become strongest at the "X" points inbetween the islands. The argument above said that largest scales grow the fastest – in this simulation the largest scales are those allowed by the size of the numerical grid. Note the velocity streamlines, showing flow patterns into and through the magnetic islands and X points.

The critical point is that this instability leads to a topological change in the field lines; it is not at all simple dissipative diffusion. The field lines reconnect across the initial neutral sheet. Some authors discuss the tearing mode as a change from one possible equilibrium state (say, an ideal one) to another (call it a resistive one). If the new, reconnected state has a lower magnetic energy, then the instability can go spontaneously; the reduced magnetic energy will appear as heat.

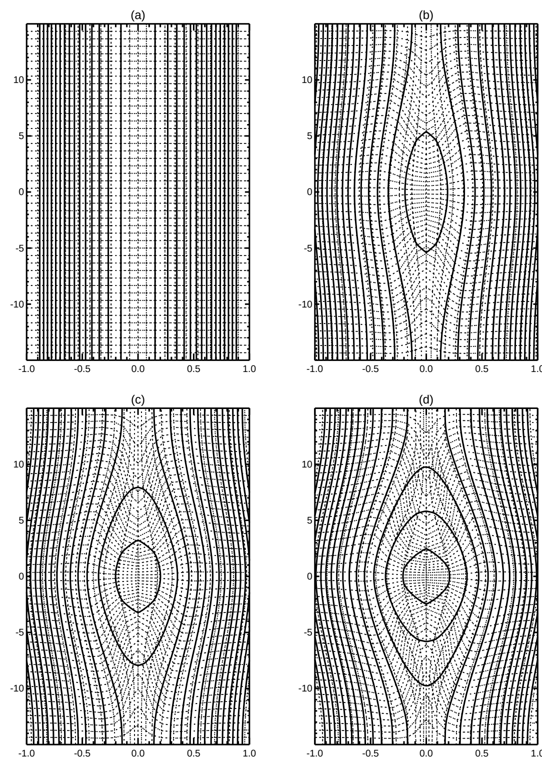

Figure 21.3. A 2D simulation illustrating the evolution of a reconnecting tearing mode. Solid lines are magnetic field lines (lines of constant flux function ψ); dotted lines show the velocity field. The initial state (top left) and three later times are shown; the formation of magnetic islands is apparent. From Garasi, PhD thesis (NMSU), 2002.

References

The classic, and nearly opaque, paper which introduced this subject is

Furth, H. P., Kileen, j. & Rosenbluth, M. N., 1963, *Phys. Fluids*, **6**, 459.

I have also followed Shavamoggi for most of the math details.