22. TURBULENCE, PART I

Turbulence is an old topic which remains fresh today. It was studied as long ago as the 16th century, when Leonardo da Vinci studied turbulence generated by obstacles placed in a water flow. It is still being studied today, when such issues as the role of small-scale vortex ropes, the effect of magnetic fields in MHD turbulence, or "quantum turbulence" in superfluids, are topics of current research. In fact, many of the basic questions are still unanswered today. Because turbulence is fundamentally nonlinear, analytic solutions are hard to come by; traditional work relies heavily on scaling laws, while large numerical simulations are critical to modern work.

To set the stage, I know of no better opening, than to paraphrase Tennekes & Lumley.

> Most flows occuring in nature and in engineering applications are turbulent. These include the boundary layer in the earth's atmosphere; jet streams in the upper troposphere; and cumulus clouds which are in turbulent motion. Subsurface ocean currents are turbulent. Stellar atmospheres are turbulent, as is the gaseous interstellar medium. Boundary layers on aircraft wings are turbulent. Most combustion processes involve turbulence. The flow of natural gas and oil in pipelines is turbulent, as is water flowing in rivers and canals. THe wakes of ships, cars and aircraft are in turbulent motion. In fluid dynamics laminar flow is the exception, not the rule: one must have small dimensions and high viscosities to encounter laminar flow.

A. The transition to turbulence

The instabilities discussed in the last chapter can develop into full-fledged turbulence. To explore this, let's choose a particular setting: consider a boundary layer in an initially laminar flow. In order to do stability analysis, we need a mathematical description of the boundary layer. Here's a sketch of one.

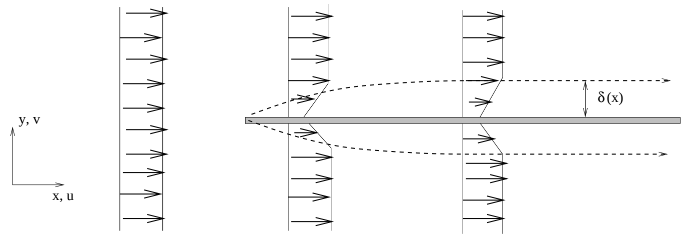

Figure 22.1. Flow past a thin plate. An initially uniform flow comes in from the left; when it encounters the plate, the no-slip requirement at the plate surface causes a viscous boundary layer to develop. The width of the layer, $\delta(x)$, grows with x in this situation.

A standard representation of such a boundary layer is the Blasius solution. (I follow Tritton, 2nd edition, chapters 8, 11). Assume incompressible flow, and let u b, the x-velocity, v be the y-velocity. The basic equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial u} = \nu\frac{\partial^2 u}{\partial x^2}\ ; \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (22.1)$$

To solve these, we do dimensional analysis (more formally, look for a similarity solution). We want to find a solution of the form $u = u_o g(y/\delta)$, if u_o is the upstream/incoming velocity, g is some unknown function, and δ can be a function of x. Because this is incompressible, we can use a stream function ψ: $u = \partial\psi/\partial y$; $v = -\partial\psi/\partial x$. We therefore look for a ψ solution with the form $\psi(x,y) = u_o\delta f(y/\delta)$; f is another ast-yet-unknown function. If we carry out the algebra and put these into the basic equations (22.1), we find (a) a new ODE; and (b) a consistency condition on δ. The results are

$$\delta(x) \propto (\nu x/u_o)^{1/2}\ ; \qquad ff'' + f''' = 0 \qquad (22.2)$$

The first is our desired result: the shape of the boundary layer (the constant of proportionality depends on just how δ is defined, for instance is it the point where $u(y) = 0.9u_o$? $0.99u_o$?). The ODE has to be solved numerically; the primes represent differentiation with respect to the similarity variable, $\eta = y/\delta(x)$.

Now, assume we have the full solution for the boundary layer (rather than just the sketch above). Subject it to the same type of analytic instability analysis as we did in chapter 19. That is, assume the perturbed (x,y) velocities, and also the perturbed pressure, go as $f(y)e^{i(kx+\omega t)}$; and determine whether the frequency has any imaginary part. If it does, the system is unstable. The key parameter here is the viscosity, ν – which is folded into the boundary-layer Reynolds number, $Re = u_o\delta/\nu$. Figure 22.2 shows a typical result: any values of (k, Re) inside the curve are unstable, while values of (k, Re) outside the curve are stable. Thus: there is a minimum Reynolds number for instability – for lower values viscosity stabilizes the flow. Above this value, there is a finite range of scales which are unstable: very small and very large scales remain stable.

What happens next? From this type of analysis, as well as from experiment, we expect that all flows become turbulent at high enough Re. Linear stability theory can tell you when it starts .. but the waves pre-

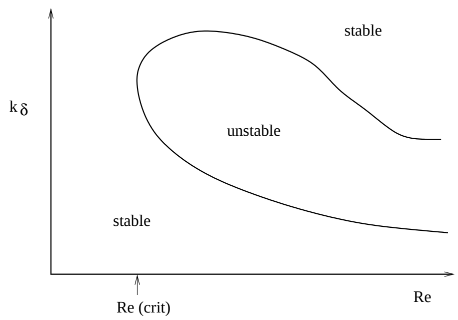

Figure 22.2. Results of stability analysis on the Blasius model of a shear layer.

dicted by such theories are only the first stage of the process. When these waves reach a critical amplitude - typically about 1% of the free-stream velocity – the nonlinear terms in the driving equations start to matter, and the flow can no longer be described by analytic means. Following Tritton, who is describing the transition to turbulence in a boundary layer:

The waves become three-dimensional, and interact with the mean flow; growth continues until there are transient local regions of high shear. Up to now, the timescale of the fluctuations has remained of the same order as the period of the initial wave; now, fluctuations with much shorter timescales appear, and develop radpily to bursts of turbulent-like fluctuations. The boundary layer changes now, develops hairpin-shaped vortices, which generate more high-shear regions – so that small areas of the boundary layer become turbulent ("turbulent spots"). These regions then grow, while they travel downstream, and eventually merge into one-another. Note that the transition is qualitatively similar in other types of flows used to study turbulence – pipe flows or free jets.

B. Turbulent flows: overview

Now that the flow has gone turbulent, what can we say about it? It can be statistically treated as a random system (*cf.*§22.3, also chapter 23); but it has some characteristic properties. Many of the authors I've looked at go into long qualitative descriptions at this point; in this section I reproduce some of their discussions.

Tritton notes that turbulence is a state of continuous instability. Each time a flow changes as a result of an instability, one's ability to predict the details of the motion is reduced. When successive instabilities have reduced the level of predictability so much that we must resort to statistics, rather than predicting the details, then we say the flow is turbulent. However, such flow is *not* compoletly random. All flows involved organized structures; and the mean values (in a statistical treatment) are still predictable. Notes, "there is every reason to suppose that this loss of predictability occurs as a property of the Navier-Stokes and continuity equations, although these equations contain the determinism of classical mechanics." It is not that the onset of turbulence represents a breakdown of these equations, but rather that the nonlinear terms allow interesting behavior!

1. CHARACTERISTICS

One can list characteristics of turbulent flows (I'm following Mathieu & Scott, also Tennekes & Lumley):

•**It is a, irregular, random process**. The flow is random, irregular, chaotic. It is time and space dependent with a very large number of degress of freedom. Although it's unpredictable in detail, its statistical properties are reproducible (so that it's deterministic in a mean sense).

•**It contains a wide range of different scales**, as can be seen from the random-looking measurements of the flow: large scales coesits with small ones (these are the "fur" in a time-series plot). Turbulent dynamics involve all scales, which coexist and are superimposed in the flow, with smaller ones living inside large ones.

•**It has small-scale random vorticit**y. It is rotational and three-dimensional, with high levels of fluctuations vorticity. The larger identifiable vortical structures are the *turbulent eddies* that are apparent in some pictures. Vortex convection and stretching may be thought of as the mechanisms by which the intense, fine-scale vorticity fluctuations are generated and maintained. Viscous diffusion causes vortices to spread, counteracting the amplifying and scale-reducing effects of stretching.

•**It arises at high Reynolds numbers**, as we saw above. As Re rises, the nonlinear terms on Navier-Stokes equation become more important, compared to the viscous term. In addition, the tendency to instability (which is damped by viscosity) increases. Once turbulence goes, flow instabilities keep it going; large-scale eddies are themselves unstable, giving rise to smaller ones, and so on, until viscosity comes in at small enough scales.

•**It dissipates energy**. The inertial range is inviscid, so it conserves mechanical energy. As smaller scales form, they can be thought of as sapping some of the KE of their parents, and transmitting it to their own offspring. Thus, a cascade arises, which generates the smaller scales, and allows a mean flux of energy from large to small scales. This flux is controlled by the dynamics of the large scales, and dissipated as heat at the smallest scales.

•**It is intrinsically 3D**. In 2D flows, there is no vortex stretching; thus, in the absence of viscous diffusion, vorticity is passively convected, unchanged by the flow. Therefore the high-Re energy cascade can't occur. 2D flows do show complicted structures, and a degree of randomness, but they don't have the ubiquitous fine scales associated with 3D turbulence. We will also see that 2D turbulence admits a reverse cascade, unlike 3D turbulence. As M&S put it, "we don't mean to insult the many authors who have talked of 2D turbulence....but to point out that the physical mechanisms are very different" in the two cases.

•**It is insensitive to viscosity** at high enough Re. That is, the dynamics of the large scales are essentially inviscid (as $\nu \to 0$); and they control the system. The size of the smallest scales adjusts to changes in ν, so as to dissipate energy at the right rate; and that rate is controlled by the large scales. From here, we go to the conjecture, that ε approaches a constant, finite value as $\nu \to 0$.

2. BEHAVIOR

Alternatively, stepping back a bit, we can describe how turbulent flows behave (Here I follow T&L):

•**Self-Organization**. Large-scale *coherent* structures can easily be identified in some flows. To be specific, consider measurements at a fixed point in the outer part of a turbulent flow. One finds periods of high frequency fluctuations, as the fixed point encounters actuve turbulence, and quiet periods, when the fixed point is in a quiescent region. This seems to be more important in 2D turbulence. The growth of these structures proceeds through vortex coalesence.

•**Entrainment** A flow can pull the surrounding fluid into itself and accelerate it along. This is called *entrainment*. We saw this process in a laminar flow (back in roblem set 1), where viscosity accounted for the entrainment. Entrainment also occurs in a turbulent flow, at a much higher rate. The connection here is inertial; think of ambient fluid which is "gulped" into the turbulent flow by the large eddies at the boundary.

•**Self-Preservation** At large downstream distances, the mean field in many shear flows becomes approximately self-similar. That is, quantities such as the central velocity are functions only of the *local* scales of length and velocity. For instance, let δ be the width of a turbulent jet (defined, for instance, as the half-power width of the cross-sectional $v(y)$ profile). Self-preservation means that quantities such as the central velocity can be written as functions of y/δ.

•**Turbulent Mixing** All transport processes can be enhanced in a turbulent situation. Diffusion coefficients (for transport of some trace substance), viscosity coefficients (transport of momentum, right?), and electrical resistivity (transport of . . .) are all enhanced. Calculating the exact transport rates involves understanding the small-scale details of the turbulence, which has rarely been fully worked through; but dimensional scaling arguments are often used.

As an example, consider turbulent viscosity ("eddy viscosity"). Kinetic theory says that the microscopic viscosity coefficient $\sim v\lambda$, where v is the mean thermal speed and λ is the mean free path (fuller analysis puts a numerical coefficient in front of this). A useful rule of thumb is that turbulent viscosity can be estimated as $\nu_t \sim v_t\lambda_t$, if v_t and λ_t are "characteristic" velocity and length scales of the turbulence. (We will specify these a bit more below).

•**Turbulent Dissipation** Viscous stresses will quickly damp out turbulence which is not continually driven. As with mixing, we can use scaling arguments to estimate the timescale for viscous damping of a turbulent flow. Free turbulence will decay on several times τ_t, if $\tau_u = \lambda_t/v_t$ is the "eddy turnover time" for the largest structures, at size λ_t and velocity v_t as above.

•**Self-Preservation** At large downstream distances, the mean field in many shear flows becomes approximately self-similar. A turbulent "free jet" (that means not bounded by a wayy nor at a boundary layer) is one example (which may appear in the homework). Flow quantities such as the central velocity are functions only of the *local* scales of length and velocity. For instance, let δ be the width of a turbulent jet (defined, for instance, as the half-power width of the cross-sectional $v(y)$ profile). Self-preservation means that quantities such as the central velocity can be written as functions of y/δ.

C. Homogeneous Turbulence

Turbulence theory was initially developed for homogeneous, isotropic turbulence. In particular, it focused on scale small compared to the overall size of the system (so that boundaries can be ignored), and considered the energy transfer between scales. This work is mainly due to Kolmogorov, and dates from the 1940's.

1. OVERVIEW

To motivate this, start with two experimental facts, which apply to fully developed (high Re, distant boundary) turbulence (following Frisch here).

• **Two-thirds law**: the rms velocity increment

$\delta v(l)^2$, between two points separated by a distance l, obeys $\delta v(l)^2 \propto l^{2/3}$. Put into wave-number space,[1] this becomes what is now known as the Kolmogorov law: $v^2(k) \propto k^{-5/3}$.

• **Finite energy dissipation**: in an experiment, vary viscosity while keeping everything else the same: find energy dissipation per unit mass behaves in a way consistent with a finite limit.. Following Frisch, think about drag. Consider a fluid moving past a rock (or a car moving through air). The drag force $F = (1/2)C_d \rho A v^2$, (you have seen this, right?) if $A = L^2$ is the projected area of the rock/car, and C_D is the coefficient of drag. Experiments show that C_D is only a very slow function of Re. Thus, the work done $W = Fv$; and the energy dissipated per unit mass is

$$\varepsilon = \frac{W}{\rho L^3} = \frac{1}{2} C_D \frac{v^3}{L} \qquad (22.3)$$

Thus, ε does not depend (very much) on ν, and thus will have a finite limit as $\nu \to 0$.

2. EDDIES AND THE ENERGY CASCADE

One critical fact about fluid turbulence is that energy is transferred from large scales to smaller scales. Picture, say, initially large eddies, on the order of the driving scale of the turbulence. This could be the width of a boundary layer, or the diameter of a pipe. Now, consider slightly smaller eddies. Due to vortex stretching, they are strained by the velocity field of the largest ones, thus growing in strength; they extract energy from the larger ones. This continues on to still smaller scales. Thus: *the turbulent kinetic energy cascades down from large to small eddies in a series of small steps.* This process is essentially inviscid, since the vortex stretching mechanism arises from the nonlinear terms in the equations of motion.

Following Tritton: the figure illustrates the evolution of a blob of fluid, due to the actions of the locally turbulent velocity field. Showing (1) the process of repeated instability; each stage gives rise to motions of greater complexity and smaller scales, than the previous stages; (2) energy is extracted from larger scales (or the mean flow), to smaller (refer ahead to mean-flow equations and Re stress); (3) vortex stretching is part of the process. The random nature of turbulent motions is diffusive, as two particles that happen to be close, at a give time, find themselves much further apart soon after. If these two particles are on the same vortex line, we get stretching – which increases the magnitude of the vorticity (wny?), but also reduces the cross-section of the vortex tube.

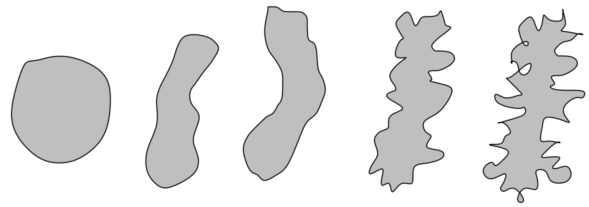

Figure 22.3. Schematic representation of the evolution of a "marked blob" of fluid within turbulent motion. Its shape becomes more and more distorted by the velocity fluctuations, with smaller and smaller scales appearing, as time goes on. Eventually the scales become small enough that diffusion matters, and the marked fluid mixes with its surroundings.

3. THE KOLMOGOROV SCALING ARGUMENTS

We can talk about the main properties and scaling laws for homogeneous, isotropic turbulence, following Kolmogorov's analysis, without needing the details of the statistics. Kolmogorov argued that properties of the flow are determined by the scale l, and the energy rate ε, *only*. (motivated by the observations, above). Then, from this, K. argued (or "derived"), the important law:

$$\delta v(l)^3 = (4/5)\varepsilon l \qquad (22.4)$$

What is the rate of energy transfer in the cascade? Let $v_t = \langle v \rangle$ be the mean turbulent speed, and let $\lambda_t = \langle \lambda \rangle$ be the largest (driving) scale.[2] The time for these largest eddies to "turn over" must be $\tau_t \sim v_t/\lambda_t$. Observations find that the large eddies transfer much of their energy to smaller ones, in one or two τ_t times. Thus the energy cascade rate, which also must become the energy dissipation rate, is

$$\varepsilon \simeq \frac{v_t^3}{\lambda_t} \qquad (22.5)$$

Kolmogorov argued that scales in this range should not be affected either by $k_t = 2\pi/\lambda_t$ (as eddies at k are driven only by their immediate neighbors in k-space), nor by $k_d = 2\pi/\lambda_d$ (basically for the same reason; energy is moving downward in k-spacd). Thus, letting

[1] How? change variables, to $k \simeq 2\pi/l$; and by energy conservation, note that we must have $v^2(l)dl = v^2(k)dk$.

[2] A formal version of this is the *Taylor microscale*, defined by some authors as $\lambda_T^2 = [v^2]/[\nabla v]^2$, and by others as $\lambda_T^{-2} = d^2C(r)/dr^2|_0$, that is the curvature of the correlation function at zero lag. Either definition has the same content: this is the scale where most of the turbulent energy is concentrated.

v_l be the velocity typical of scale l, the energy flow rate $\varepsilon = v_l^3/l$ must be independent of the scale l (or $k = 2\pi/l$). What power spectrum $W(k)$ is consistent with this picture? The constancy of ε tells us that

$$v_l \simeq (\varepsilon l)^{1/3} \; ; \qquad v_k \simeq \left(\frac{2\pi\varepsilon}{k}\right)^{1/3} \tag{22.6}$$

But also, the power "at k" is $v_k^2 \simeq kW(k)$. Thus we find the equilibrium spectrum,

$$W(k) \propto \varepsilon^{2/3} k^{-5/3} \tag{22.7}$$

This is the *Kolmogorov spectrum*; and the range of wavenumbers to which it applies is called the *inertial range*: $k_t \ll k \ll k_d$.

Where does this cascade end? It cannot to on to infinitely small scales. In fact, Kolmorogov suggested that the smallest scales to which the cascade reaches are those on which the viscous dissipation rate is equal to the energy cascade rate. But we can get this by dimensional analysis. The units of ε are cm^2/s^3, and those of viscosity are cm^2/s. Thus, the *dissipation scale* must be

$$\lambda_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \tag{22.8}$$

(Some authors call λ_t the *outer scale*, and λ_d the *inner scale*.) Note that the ratio of the inner and outer scales just depends on the Reynolds number[3]: $\lambda_t/\lambda_d \sim Re^{3/4}$.

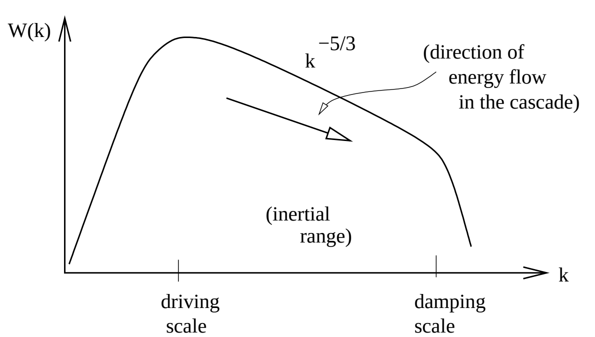

Figure 22.4. Sketch of the energy spectrum in Kolmogorov turbulence. Energy comes in at the driving scale (think of the radius of a turbulent jet, or the width of the bounday layer). It cascades forward due to wave-wave interactions (or vortex stretching) at each intermediate scale k; and dissipates by small-scale viscous damping at the damping scale.

[3] Check: can you show this? You need to define Re in terms of the turbulent quantities v_t and λ_t.

Finally, it's worth noting that we can also find the dissipation scale by noting that the *local* energy transfer rate, v_l^3/l, must equal the energy dissipation rate, $\nu v_l^2/l^2$, on this scale. Equating these two recovers (22.8) nicely.

Below k_t the power falls off, due to a lack of driving, and the fact that turbulent energy only cascades forward in hydrodynamic turbulence. Above k_d the power spectrum is usually taken to fall off more steeply, ad $W(k) \propto k^{-3}$. The data agree with this very well (lots of people have looked at this).

4. WHAT IF THE FLUID IS MAGNETIZED?

Just a note here looking ahead. Everything we've seen in this chapter is well understood and well supported by data. Isotropic turbulence in non-MHD fluids is very well described by the Kolmogorov model. The situation changes dramatically, however, when the fluid is magnetized. We'll talk about this a bit in chapter 24.

References

Qualitative descriptions of turbulence can be found in many books, as can the Kolmogorov scaling arguments. I've mostly followed Tennekes & Lumley, and Tritton, who present the traditional work. Frisch's book is also of some use, and goes into some more modern work; Hinze is one good source for the more mathematical (Fourier space) treatments.

D. Appendix: fun facts from Fourier transforms

To work in any detail with turbulence, we need some statistical tools, from the theory of random variables. Say we measure some property of the turbulence – the velocity in some direction, say – either at one point, as a function of time, or instantaneously (don't ask me how!) as a function of position. We will treat the velocity $v(x)$ as a random function of position and assume that the time measurement and the space measurement give us the same information. Let the turbulent velocity field (continue letting v be 1D for now) be $v(\mathbf{x})$, and let it have a Fourier transform

$$\tilde{v}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int v(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \tag{22.9}$$

with inversion

$$v(\mathbf{x}) = \int \tilde{v}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \tag{22.10}$$

This FT, $\tilde{v}(\mathbf{k})$, contains information on "where the power is" in the turbulent signal. Note $\tilde{v}(\mathbf{k})$ can be complex. This is generally described *via* the *power spectrum* of the turbulent velocity:

$$P(\mathbf{k}) = \frac{(2\pi)^3}{V} \tilde{v}(\mathbf{k}) \tilde{v}(\mathbf{k})^* = \frac{(2\pi)^3}{V} |\tilde{v}(\mathbf{k})|^2 \tag{22.11}$$

(the normalization is somewhat arbitrary – this notation is strictly interpreted to mean the limit as the source volume, V, becomes very large).[4]

Now, consider also the *correlation function* of the turbulence:

$$C(\mathbf{r}) = \langle v(\mathbf{x}) v(\mathbf{x}+\mathbf{r}) \rangle \tag{22.12}$$

where the brackets denote a mean over the turbulent volume. This also tells us "where the power is"; we expect $C(\mathbf{r}) \to 0$ for scales $\mathbf{r}$ which are larger than any correlation length of the turbulence.[5] The correlation function can also be Fourier transformed:

$$C(\mathbf{r}) = \int \tilde{C}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \tag{22.14}$$

so that $\tilde{c}(\mathbf{k})$ is the FT of $C(\mathbf{r})$. Now, a nice result from Fourier theory is that $\tilde{C}(\mathbf{k})$ and $P(\mathbf{k})$ are related:

$$P(\mathbf{k}) = \tilde{C}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int C(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \tag{22.15}$$

A nice corrolary of this, is that the autocorrelation function at zero lag is directly related to the power spectrum:

$$C(0) = \int P(\mathbf{k}) d\mathbf{k} \tag{22.16}$$

[4] Note, in going from $\tilde{v}(\mathbf{k})$ to $P(\mathbf{k})$ we have lost information on the relative phase of the $\mathbf{k}$th mode. This is assumed not to be important in the usual homogeneous turbulence models, but phase information does matter in some applications, such as intermittency.

[5] Some authors use the *structure function:*

$$\begin{aligned} D(\mathbf{r}) &= \langle [v(\mathbf{x}+\mathbf{r}) - v(\mathbf{x})]^2 \rangle \\ &= \langle v(\mathbf{x}+\mathbf{r})^2 \rangle + \langle v(\mathbf{x})^2 \rangle - 2\langle v(\mathbf{x}) v(\mathbf{x}+\mathbf{r}) \rangle \end{aligned} \tag{22.13}$$

which is clearly a close cousin of the correlation function $C(\mathbf{r})$

$$D(\mathbf{r}) = 2\left[C(0) - C(\mathbf{r})\right] \; ; \qquad C(\mathbf{r}) + \frac{1}{2} D(\mathbf{r}) = \frac{1}{2} D(\infty)$$

Thus: the Fourier transform of the autocorrelation function of the turbulence, gives us the power spectrum of the turbulence. It can be interpreted nicely in that $P(\mathbf{k})$ describes how much energy "is contained in waves at $\mathbf{k}$". Many authors work in the limit of isotropic turbulence: $C(0) = \int 4\pi k^2 P(k)$. Thus, the isotropic spectrum is often quoted: $W(k) = 4\pi k^2 P(k)$. This is often the quantity that is addressed in Kolomogorov-type modelling.