23. TURBULENCE, PART II

In the last chapter we talked about the basic nature of turbulence: what it's like and how it's described statistically. In this chapter I carry on with more "classical turbulence", namely the effect of turbulent stresses on the mean-field flow; and say a little about "modern turbulence", including how 2D turbulence is different, and turbulence on small scales (which includes vorticity & intermittency).

A. Mean Field Equations

In many systems, we can treat the large-scale flow as steady, or at least slowly varying, with the turbulence as a rapidly fluctuating additive term. That is: let $\mathbf{V}$ be the mean velocity, and $\mathbf{v}$ the turbulent one, so that the net velocity field is $\mathbf{V}+\mathbf{v}$. Treat the pressure field similarly, $P+p$; ditto for temperature. (As long as we stay in the incompressible limit – which is where most turbulence analysis stays – there are no density fluctuations, right?) If these sums are put into the basic dynamical equations, we can (borrowing terminology from MHD turbulence), isolate the dynamical equations for the "mean field" quantities, and find how the turbulence affects the mean flow. We will find, for instance, that the mean-flow momentum equation contains what are called Reynolds stresses: non-zero terms involving second moments of the fluctuating velocity field. Thus, the mean flow and turbulence are intimately connected, with the one affecting the other.

1. THE CONTINUITY EQUATION

For instance, take the incompressible continuity equation:

$$\nabla \cdot (\mathbf{V}+\mathbf{v}) = \nabla \cdot \mathbf{V} + \nabla \cdot \mathbf{v} = 0 \tag{23.1}$$

Now, take means: that means ensemble averages, or time averages for the case of stationary flows. This gives

$$\nabla \cdot \langle \mathbf{V} \rangle + \nabla \cdot \langle \mathbf{v} \rangle = 0 \tag{23.2}$$

But now: first, we have $\langle \mathbf{V} \rangle = \mathbf{V}$; that is the "mean flow" velocity. And, we assume the turbulent fluctuations have zero mean: $\langle \mathbf{v} \rangle = 0$. Thus, (23.2) becomes

$$\nabla \cdot \mathbf{V} = 0 \tag{23.3}$$

(recovering incompressibility of our mean state). And, subtracting (23.2) from (23.1) gives

$$\nabla \cdot \mathbf{v} = 0 \tag{23.4}$$

showing that our turbulent field is also incompressible (by itself).

2. THE MEAN MOMENTUM EQUATION

This analysis can be repeated for the other basic equations; I'm not going to write down all the steps, however. The momentum equation starts as

$$\frac{\partial}{\partial t}(\mathbf{V}+\mathbf{v}) + (\mathbf{V}+\mathbf{v})\cdot\nabla(\mathbf{V}+\mathbf{v}) = -\frac{1}{\rho}\nabla(P+p) + \nu\nabla^2(\mathbf{V}+\mathbf{v}) \tag{23.5}$$

The mean of this equation becomes – written out in Cartesian

$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\left(\frac{\partial^2 V_i}{\partial x_i^2} + \frac{\partial^2 V_i}{\partial x_j^2}\right) - \frac{\partial}{\partial x_j}\langle v_i v_j \rangle \tag{23.6}$$

Thus: the turbulent field contributes a net reaction back on the mean field. This last term is called the *Reynolds stress*. Even though $\langle v_i \rangle = 0$ for each component i, the mean of their product does not necessarily vanish: $\langle v_i v_j \rangle \neq 0$. It turns out that this is so for anisotropic turbulence – and real turbulence is commonly anisotropic.[1]

Note that the Reynolds stress arisews from averaging the nonlinear convective term in the full Navier-Stokes equation; it is not truly a stress term, more like the mean momentum flux due to the turbulent fluctuations. Nonetheless, it is often talked about in terms of a *turbulent viscosity*. Because this extra force term in the basic momentum equation is the gradient of a tensor, it is generally combined with the viscous stress tensor, as

$$\sigma_{ij} = -P\delta_{ij} + \rho\nu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) - \rho\langle v_i v_j \rangle \tag{23.7}$$

We can go further, and write the turbulent contribution to the viscous stress tensor as

$$\sigma_{ij,\mathrm{turb}} = -\rho\langle v_i v_j \rangle = \rho\nu_t\left(\frac{dV_i}{\partial x_j} + \frac{dV_j}{\partial x_i}\right) \tag{23.8}$$

This last step is an assumption which turns out to be quite well justified. That is, the Reynolds stress is linearly proportional to the mean flow shears. But then,

[1] This is the case because, microscopically, viscosity mixes the relative phases of each v_i component; they do not stay in phase, and thus the mean of their product is non-zero. Alternatively: think about how Brownian motion, on the microscopic level, transports momentum components laterally.. this gives us the usual macro stress tensor. Here, the turbulent fluctuations provide a mean transport of momentum...

we can see from this that the turbulent viscosity coefficient $\nu_t \sim v_t L$, if L is the local gradient scale. This connects back to our dimensional argument, above.

3. EXAMPLE: 2D CHANNEL FLOW

I follow Mathieu & Scott here. Consider a channel in the x-direction, with y the transverse coordinate. (Once again, compare this to the laminar flows in chapter 2). Assume the mean flow independent of z, and that $V_z = 0$. Note, this does not imply that the flow is fully 2D, indeed the turbulence will be 3D. Rather, we're assusming that it's 2D in the mean, and all statistical properties are unchanged under reflection about the $z = 0$ plane. As before, put the overall mean flow in x direction, and let it be driven by some dP/dx. Also as before, all quantites except the pressure depend only on y; we extend this here to include Re stresses.

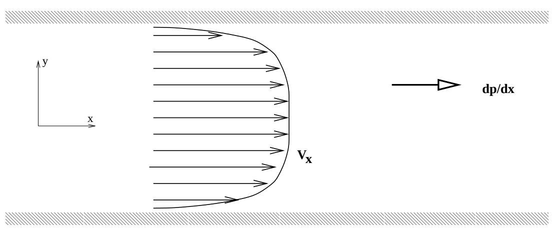

Figure 23.1. Geometry and solution of turbulent channel flow. The flow is symmetric about the midplane, $y = 0$; the walls are at $y = \pm D$. Following Mathieu & Scott Figure 4.3.

Now: the assumed symmetry in the z direction means only the offdiagonal term $\langle v_x v_y \rangle$ in the Re stress will be nonzero; and it depends only on y. Thus, the mean force equation has two nonzero components:

$$\begin{aligned} \frac{\partial P}{\partial y} + \rho \frac{d\langle v_y^2 \rangle}{dy} &= 0 \\ \nu \frac{d^2 V_x}{dy^2} = \frac{\partial P}{\partial x} &+ \rho \frac{d\langle v_x v_y \rangle}{dy} \end{aligned} \tag{23.9}$$

From the first of these, we get $P(x,y) = P_w(x) - \rho\langle v_y^2 \rangle$: where $P_w(x)$ is the pressure at the wall (where $v_y = 0$ by no-slip); and the second term is the diagonal contribution of the Re stress to the net pressure. Putting this into the second equation in (23.9), we get

$$\frac{dP_w}{dx} = \frac{d\tau}{dy} = \frac{d}{dy}\left[\nu \frac{\partial V_x}{\partial y} - \rho \langle v_x v_y \rangle \right] \tag{23.10}$$

The total mean stress is defined as $\tau = -\rho\langle v_x v_y \rangle + \nu dV_x/dy$. But now: the left hand side of (23.10) is only a function of x, while the right hand side is only a function of y; thus, each side is a constant. The left hand side is just the driving pressure gradient. From the constancy of τ we get $\tau = y dP_w/dx + \tau_w$ and τ_w is another constant, the mean viscous shear stress at the wall $y = 0$. We find its value by noting that the flow must be symmetric about the midplane; thus $\tau_w = -D dP_w/dx$. Using this in the definition of τ, we get

$$\tau_w \left(1 - \frac{y}{D}\right) + \rho\langle v_x v_y \rangle = \nu \frac{dV_x}{dy} \tag{23.11}$$

This is almost the answer for the mean flow .... as we're turbulent, we know $Re \gg 0$, so that the RHS is small except very close to the walls. Thus, away from the boundary layer, $-\rho\langle v_x v_y \rangle \simeq \tau_w(1 - y/D)$, that is linear behavior. And, if this balance holds, we expect V_x to be independent of y; and "measurements find V_x is approximately constant in this region". Near the wall, we expect steep gradients in V_x, so that viscosity matters...and a more complicated behavior of the Reynolds stress.

4. THE MEAN ENERGY EQUATION

Finally, the mean-field energy equation is worth noting. IF we multiply the mean-field momentum equation by V_i and sum on i, we get

$$\frac{1}{2}\frac{dV_i^2}{\partial t} + \frac{1}{2}V_j \frac{\partial V_i^2}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(V_i \tau_{ij}\right) - \frac{1}{\rho}\tau_{ij}\frac{\partial V_i}{\partial x_j} \tag{23.12}$$

Expanding the τ_{ij} term back out, defining

$$E_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{23.13}$$

and noting that $\delta_{ij}\partial V_i/\partial x_j = \partial V_i/\partial x_i = 0$, we have a useful form:

$$\begin{aligned} \frac{D}{Dt}\left(\frac{1}{2}V_i^2\right) =& \frac{\partial}{\partial x_j}\left(-\frac{PV_j}{\rho} + 2\nu V_i E_{ij} - \langle v_i v_j \rangle V_i\right) \\ & - 2\nu E_{ij}E_{ij} + \langle v_i v_j \rangle \frac{\partial V_i}{\partial x_j} \end{aligned} \tag{23.14}$$

Here, the first 3 terms (inside the brackets) are advective energy transport; the term with ν is viscous dissipation; and the last term is energy lost (in a decelerating flow) to drive the turbulence.

5. WHAT ABOUT THE TURBULENT TERMS?

In these notes I've said nothing about the dynamics of the turbulent terms ($p, \mathbf{v}$, etc). That was on purpose .... refer back to the scale separation we used in deriving the mean-field equations (23.3), (23.6), (23.14). We could equally well have separated out the D/Dt

terms involving the turbulent fields. Remember, now, that flow equations are nonlinear (the $\mathbf{v} \cdot \nabla \mathbf{v}$ term, for instance). When one isolates the dynamical equations for the fluctuations, one finds higher-order terms are involved. For instance, the equation for $D\langle v_i \rangle / Dt$ involves $\partial \langle v_i v_k \rangle / \partial x_k$; the equation for $D\langle v_i v_j \rangle / Dt$ involves $\partial \langle v_i v_j v_k \rangle / \partial x_k$; and so on. Some additional assumption is always needed to close a system like this; and pursuing that would take us too far afield.

B. Two-dimensional Turbulence

. . . differs significantly from turbulence in three-dimensions. The most striking difference is that turbulent power in 2D cascades both forward (to higher wavenumber) and backwards (to lower wavenumber). This is still a topic of active discussion in the literature. I mostly follow Biskamp's discussion in these notes.

Recall 3D turbulence: we saw that turbulent power undergoes only a forward cascade. We also recall that three-dimensional fluid flow has two significant invariants:

$$\mathcal{E} = \frac{1}{2} \int v^2 dV, \text{ energy} \tag{23.15}$$

and

$$\mathcal{H}_V = \frac{1}{2} \int \mathbf{v} \cdot \nabla \times \mathbf{v} dV, \text{ velocity helicity} \tag{23.16}$$

(Invariance of the first should be obvious; invariance of the second may be proved in the homework.[2]) Both energy and velocity helicity are conserved in the mode-mode interactions which set up the cascade. Such small-scale interactions drive both $\mathcal{E}$ and $\mathcal{H}_V$ forward, to smaller wavenumbers. The second invariant in 2D turbulence, however, is different. The two invariants in 2D are

$$\mathcal{E} = \frac{1}{2} \int v^2 dV, \text{ energy} \tag{23.17}$$

and

$$\Omega = \frac{1}{2} \int \omega^2 dV, \text{ enstrophy} \tag{23.18}$$

(The enstrophy is a fancy name that turbulence types like to use for the mean square vorticity.) In 2D turbulence, wave-wave interactions drive Ω forward, but drives $\mathcal{E}$ to smaller wavenumbers – in an *inverse cascade*. This cascade drives power to larger and larger scales: one author described it as the buildup of coherent vortices, in which nonlinear distortions nearly vanish, and which continue to grow by vortex coalesence.

One can use arguments similar to those in §22.3 to predict the turbulent spectrum. Let k_o be the driving wavenumber, and assume it is well between the system scale k_s, and the dissipation scale k_d. Energy cascades to lower wavenumbers; the arguments of §22.3 still apply, so that a power spectrum $W(k) \propto k^{-5/3}$ should obtain for $k_{min} < k < k_o$, where k_{min} corresponds to some largest scale reached by the reverse cascade. Above k_o, the enstrophy cascade rules. Describing enstrophy and turnover time at scale l as $\omega_l \sim v_l^2/l_2$, $\tau_l \sim l/v_l$, we get the enstrophy power to be $\epsilon_l \sim \Omega_l/\tau_l \sim v_l^3/l^3$. But this last must be independent of l (repeating the Kolmogorov argument); from this we need $v_l \propto l$, so that $kW(k) \propto v_k^2 \propto k^{-2}$. This should describe the cascade up to the dissipation range. Thus, for 2D fluid turbulence driven at k_o, we expect

$$\begin{aligned} W(k) &\propto k^{-5/3}\ ; \qquad k_{min} < k < k_o \\ W(k) &\propto k^{-3}\ ; \qquad k_o < k < k_d \end{aligned} \tag{23.19}$$

This is supported by observations. As in 3D turbulence, the high-k part of the cascade can be steady-state. Energy input to the system at k_o will cascade forward, reaching the dissipation scale k_{diss} where it goes into hear. On the contrary, however, the low-k part of the cascade cannot be stable. There is no low-k dissipation to balance the energy input. One would expect k_{min} to decrease with time, until the system size k_s is reached; at and after this point energy will continue to accumulate at the largest scales allowed in the system.

What sets the cascade direction? I have not found any clear answer in the literature. On small scales, the nature of energy, enstrophy and helicity transfer in mode-mode interactions (say, $\mathbf{k}_1 + \mathbf{k}_2 \rightarrow \mathbf{k}_3$) must have a preference for forward or reverse transfer. The details of these processes seem not to be obvious, however. Biskamp reports on work in the literature addressing the (thermodynamic) equilibrium distribution of $\mathcal{E}(k)$, $\mathcal{H}_V(k)$, and $\Omega(k)$. It appears that the equilibrium distributions of both $\mathcal{E}(k)$ and $\mathcal{H}_V(k)$ are both weighted towards high k's in 3D; while in 2D the distribution of $\Omega(k)$ is weighted to high k, but that of $\mathcal{E}(k)$ is weighted to low k. The inference, then, is that the cascade direction is set by the tendency of the system to move towards a statistical equibrium state.

[2] The methods are similar to the proof that magnetic helicity is invariant, back in chapter 15.

C. Small scales and intermittency

Much of the action these days seems to be "what's happening on small scales", which means close to the Kolmogorov dissipation scale. This is potentially a big topic; I'm storing only a brief overview here.

1. INTERMITTENCY

The term intermittency is used in two different ways — which drove me crazy when I was trying to learn this field.

Older: macroscopic intermittency. Turbulence observed on large scales (comparable to the system size, well above the dissipation range) is intermittent: only a fraction of the volume is filled with turbulent spots or eddies at any instant. Sit at one point in a turbulent region .. you will find periods of high frequency fluctuations, and also quiet periods, as the turbulent patches evolve into/out of your region. This seems to be how the term is used in the older literature.

Newer: microscopic intermittency. Turbulence shows quite interesting behavior when observed on scales comparable to the dissipation scale. On these small scales, you find intermittent random "bursts" of energy (think of measuring the velocity as a function of time); this which translates into non-Gaussian statistics, which many authors use as a definition of intermittency. Physically, careful observations and numerical simulation reveal the existence of a tangle of intense, slender vortex ropes on these small scales. This is the way the term is used in the current literature.

2. TURBULENCE ON SMALL SCALES

The current discussion on small-scale structure is quite interesting. Problems with the Kolmogorov picture (called K41) on small scales seem to have been realized early on. It was noted that velocity derivatives did vary with the Reynolds number (which is inconsistent with K41 assumptions). Batchelor and Townsend interpreted this as "a tendency to form isolated regions of concentrated vorticity". Moffatt reports a comment from Landau to Kolmogorov, that in local regions of higher ε, the cascade will proceed more vigorously. Thus an intermittent distribution of $\varepsilon(\mathbf{x}, t)$ is expected; and this should affect the 5/3 exponent slightly, but should have stronger effect on higher-order statistical quantities (such as the velocity derivative).

This can also be interpreted statistically. A central assumption of the Kolmogorov theory is the self-similarity of the random (velocity) field at all scales. Therefore, you should find the same statistics on all time/space scales. This turns out not to be so: on small enough scales (high enough frequencies, think of a high-pass filter), you find intermittent "bursts" of energy. Connect this to statistics: a self-similar random signal, which is the same on different scales, is a white noise signal, and has Gaussian statistics. When you go intermittent, you have a greater chance of getting large bursts, and less chance of getting low-amplitude signal .. so the probability distribution function (PDF) flattens. Several authors show PDF plots with tails much flatter than Gaussian. This becomes conspicuous only on scales comparable to, or smaller than, the dissipation scale. Thus, it is characteristic of the dissipation range, and does not imply breakdown of the entire K41 analysis.

3. THE ROLE OF VORTEX FILAMENTS

One more commenet on the topic of statistics: the statistical approach we introduced in §22.C was based on the power spectrum of two-point correlations. It threw away phase information about the Fourier components of the flow, as well as any information about higher-order correlations. In particular, that means that the power-spectrum approach can't say anything about locally anisotropic structures, such as small-scale vortex ropes. But we are learning, more and more, that vorticity is important on small scales. So, we can't just rely on analytic methods, let alone scaling arguments. We need to turn to numerical simulations – which have made great strides in recent years, as computers get faster and sophisticated numerical codes are developed to take advantage of them. We can also turn to careful experiments, designed to probe small scales.

The picture emerging from simulations and experiment is as follows. Essentially all simulations show persistent and extended tubes, sheets and blobs of small-scale vorticity. The filaments are tubes with approximately circular cross section, and diameter on the order of the dissipation scale. Their length is somewhere between the Taylor scale and the driving (outer) scale. The internal structure and dynamics of these little vortices is still unclear, it seems – attempts to use analytic models (such as the Hill's vortex we saw in homework, or a linear vortex model called Burgers) have not been all that successful. There is definitely a sense in the literature that these dissipation-scale vortes tubes play an important role in the overall structure and dynamics of turbulence ... but, again, just how that works is far from clear. So this field is still evolving. Meanwhile, I like the quote from Moffatt, Kida and Ohkitani (1994):

> Just as sinews serve to connect a muscle with a bone or other structure, so the concentrated vortices of turbulence serve to connect large eddies of much weaker vorticity; and just as sinews can take the stress and strain of muscular effort, so the concentrated vortices can accomodate the stress associated with the low pressure in their cores and the stress imposed by relative motion of the eddies into which they must merge at their ends.

References

The mean-field material is “traditional”, and can be found in various books which treat turbulence mathematically. Tennekes & Lumley, or Hinze, are good sources.

The newer turbulence material, especially on intermittency, I’ve taken from “here and there”. The *Annual Review of Fluid Mech* has several useful reviews (including the Moffat homage-to-Batchelor article from 2003).