24. MHD TURBULENCE AND DYNAMOS

MHD turbulence can be quite different from the fluid turbulence we considered in chapters 22 and 23. The study of MHD turbulence is a much newer field, with must less experimental support or motivation. Many of the arguments and conclusions appear to be less "engraved in stone" than is the case for fluid turbulence. This doesn't mean the physics is unimportant, though. A critical application is to magnetic dynamos, which are clearly important (what maintains the earth's B field? The sun's? What makes each one reverse every so often?), but not yet understood. We do know, however, that naturally occuring (plasma/MHD) dynamos could not exist without turbulence. So we need to understand the one, in order to understand the other.

A. Magnetic fields in Isotropic turbulence

To start, let's think about the effects of isotropic (mainly HD) turbulence on a passive, "isotropic" B field (one that's initially very small, with no imposed order, and thus dynamically unimportant). The most basic fact is that the turbulence will amplify the field: as the plasma flows stretch and twist the field lines, the magnitude of B will increase with time. We might guess that the field will grow to "energy equipartion" with the turbulence, *i.e.* $B^2/8\pi \lesssim \rho v_t^2/2$, before it can "fight back" against the flow and make the whole problem more complicated. Simulations show that this energy balance is more or less true. But what is the structure of the B field? Does it have power on large or small scales?

To approach this, you may remember that we noted the similarity between the dynamical equation for vorticity (4.8), and that for the B field (13.7). We know that HD turbulence contains – or is held together – by small-scale vortex ropes, which turn up around the dissipation scale. We therefore expect MHD turbulence to show concentrated magnetic flux ropes on small scales. This is somewhat of a new area, but there is suggestive evidence from numerical simulations that this is the case. Both analytic estimates and simulations – mostly the latter – now show that the the magnetic field in isotropic, fluid turbulence tends to be concentrated into small-scale filaments. We can, in fact, guesstimate the size of the flux ropes. Remember the magnetic Reynolds number, $\mathrm{Rm} = v_t \lambda_t/\eta$ (defined here in terms of turbulent velocity and length scales). These filaments have length $\sim \lambda_t$, and thickness $\sim \lambda_t/\mathrm{Rm}^{1/2}$. Some simulations also find that most of the magnetic energy exists around these scales: there isn't much large-scale B field in this situation.[1]

On magnetic flux ropes ... we know from astrophysics that they are common (perhaps "B fields like to clump into flux ropes"?). One example is the B field at the surface of the sun is concentrated in intense, narrow flux ropes – the locations where they emerge through the surface are sunspots. There are hints elsewhere in astrophysical data – observations of the interstellar medium or the plasma in galaxy clusters – which also point to the existence of magnetic filaments on these scales.

B. The Inertial Range in MHD Turbulence

How does this change if there is an external, ordered $\mathbf{B}$ imposed on the system to start – even if it's quite weak? How, if at all, does a magnetic field change Kolmogorov's arguments about the energy casacade and the existence or spectrum of the inertial range? That's not yet clear. Everyone agrees that energy still cascades forward, to smaller scales, in MHD turbulence. But just how that happens is a matter of ongoing debate.

To be specific, assume the plasma is threaded by a straight, uniform $\mathbf{B}$ field. Let's also assume the driving is fairly weak, so that the "turbulence" itself is fairly weak. In this case, several authors have argued that the turbulence can be thought of as a field of small-amplitude wavelike structures (think of Alfven waves, or wave packets) propagating along $\mathbf{B}$. This changes the physics that governs energy transfer in the cascade, and thus impacts the nature of the internal range (compared to simple HD turbulence). Can we come up with scaling arguments to describe the intertial range, for turbulence in this limit? To proceed, we need to note that there are two critical differences betwen this case and the simpler HD turbulence described by Kolmogorov.

The first critical difference is the timescale for energy exchange between scales. Recall for HD turbulence the timescale is just the eddy turnover time at l, namely $\tau_l \sim l/v_l$. For MHD turbulence, there is a second important timescale, namely the time it takes an Alfven wave to move through the "eddy". Which of these, if either, should we take as the energy transfer time? The literature contains two important alternative possibilities, as follows.

[1] We'll see below that if the turbulence has a particular type of anisotropy – helicity – then the B field it generates has substantial large-scale energy. This is one form of a dynamo.

1. THE KRAICHNAN MODEL

The first approach comes from the 1960's. It is generally attributed to Kraichnan, but Iroshnikov published the same general idea in the Russian literature a few years earlier. So this is now called *Iroshnikov-Kraichnan* (IK) turbulence.

Following them, assume the main turbulent structures – call them "turbulent eddies" or "Alfven wave packets" are isotropic, with scale l. The Alfven time is then $\tau_A \sim l/v_A$. Now, if the turbulence is weak, v_l is small, so that $\tau_A \ll \tau_l = l/v_l$. In addition, the distortion a wave packet suffers[2] in one crossing is small, $\delta v/v \sim \tau_A/\tau_l$. The number of "hits" required to distort the packet significantly, and thus to transfer energy to another packet (as needed for the cascade, is therefore $\sim (v/\delta v)^2$. IK therefore argued that the energy transfer time in MHD turbulence is more like

$$\tau_E \sim \frac{\tau_l^2}{\tau_A} \tag{24.1}$$

This has consequences for the MHD turbulent spectrum. The energy transfer rate is now

$$\varepsilon \sim \frac{v_l^4 \tau_A}{l^2} \sim \frac{v_l^4}{l v_A} \tag{24.2}$$

From here, repeating the arguments in §22.3, we find the *Kraichnan spectrum* for MHD turbulence:

$$W(k) \simeq (\varepsilon v_A)^{1/2}\, k^{-3/2} \tag{24.3}$$

which is flatter than the Kolmogorov spectrum. This result should hold for 2D and 3D MHD turbulence (as both show a forward energy cascade).

2. ANISOTROPY; GOLDREICH-SRIDHAR MODEL

But there's another critical difference between HD and MHD turbulence. MHD turbulence is very likely to be *anisotropic*: the turbulent wave packets have two dimensions, $l_\parallel$ and $l_\perp$ (along and across $\mathbf{B}$). Now, Alfven waves will travel along the field, so we can define $\tau_A \sim l_\parallel/v_A$. But turbulent "strains" should act across the packet; so we expect $\tau_l \sim l_\perp/v_l$. If we retain the weak-eddy-interaction picture, from (24.1), we come up with a more compicated "energy transfer time:"

$$\tau_E \sim \frac{\tau_l^2}{\tau_A} \sim \frac{l_\perp^2 v_A}{l_\parallel v_l^2} \tag{24.4}$$

[2] The distortion can be measured by the fluctuation in the (transverse) plasma velocity, δv

What now? How do we deal with this?

One possibility is to argue that the A wave packets are very weak, so that $l_\parallel$ is not affected at all by interactions between them – thus we can treat it as a constant (governed by the driving). If this is the case, then only the $l_\perp \sim 1/k_\perp$ term in (24.4) is important to the cascade, and we expect

$$W(k) \sim \left(\epsilon k_\parallel v_A\right)^{1/2} k_\perp^{-2} \tag{24.5}$$

But that's not the whole story. GS suggested that this argument only works for the larger scales in the system. The "transition" spectrum, (24.5), says that less and less energy resides in the perpendicular component (of a turbulent "eddy"), while the parallel remains unchanged. Thus, the turbulent structures become more and more elongated as one goes to smaller and smaller scales. So, GS argue that at high enough wavenumber ($k_\perp$), the system will reach a state in which $v_\perp(l_\perp)/l_\perp \sim k_\parallel v_A$. This is a state in which the parallel Alfven wave transit time through the eddy is equal to the perpendicular straining time (or turnover time) of the eddy: $\tau_A \sim \tau_l$ will arise naturally, according to GS. This state is called *critical balance*. But look back at (24.1): when critical balance holds, we have $\tau_E \sim l/v_l \sim \tau_l$! That is, we now have only one time scale – and so we recover the original IK turbulent spectrum, with an additional prediction on the aspect ratio of the turbulent eddies:

$$W(k_\perp) \sim \epsilon^{2/3} k_\perp^{-5/3}\ ; \quad k_\parallel \sim \frac{k_\perp^{2/3}\epsilon^{1/3}}{v_A} \tag{24.6}$$

3. WHAT NOW?

Where do we stand today? Rather in the middle of a muddle ... there is still a lot of discussion, or argument, about which of these models works, if so where (under what conditions), or do they work at all? Maybe IK works here, GS works there? From what I see in the literature, the question is far from settled. Simulations are very valuable tools, but different simulations can give different answers (and it may be that the different physical/boundary conditions used in different simulations truly give different turbulence physics).

What about observations? Those are also hard to come by. Turbulence in the solar wind (which we can measure directly, with satellite probes, at least close to earth) was initially reported to have an IK spectrum, $W(k) \propto k^{-3/2}$ (over a few decades – 2 or 3 I think – in k). This was taken as strong support for the IK model. More recently, however, more data have been acquired. It is now established that solar wind turbulence is anisotropic; that's good, because the wind definitely has an ordered background $\mathbf{B}$. Unfortunately, however, newer data don't clearly support the IK spectrum – they suggest a steeper spectrum, maybe closer to $k^{-5/3}$. I personally suspect that we just don't know, yet.

Moving to a larger playing field, there is a much-quoted paper (Armstrong etal ApJ 1995) which attempted to pull together several different types of measurement – all indirect, of necessity – of electron density fluctuations in the interstellar medium (ISM). I have to point out that the ISM is very definitely a magnetized plasma, but also very definitely highly inhomogeneous – different experiments probe different regions and scales of the galaxy. This paper patched all the measurements together and concluded that the ISM obeys a Kolmogorov-type scaling, $W(k) \propto k^{-5/3}$, over $12(!!)$ orders of magnitude in k. I remain personally skeptical of this result, because of the difficulties and uncertainties in combining these disparate types of measurement.

C. Cascades and Related Things

In addition to MHD effects on the inertial range, there are several other important differences.

1. IDEAL INVARIANTS

How to HD and MHD turbulence compare, in 2D and 3D? What are the important invariants?

We saw in Chapter 23 that magnetic helicity is invariant in MHD flows; as long as dissipation is small (for scales above the dissipation range), total energy is as well. This analysis can be carried out for HD and MHD turbulence. One finds the following invariants.

• 3D HD: total energy $\mathcal{E} = \frac{1}{2}\int v^2 dV$; velocity helicity $\mathcal{H}_V = \frac{1}{2}\int \mathbf{v}\cdot\nabla\times\mathbf{v}dV$

• 2D HD: total energy $\mathcal{E}$; and total vorticity, also called *enstrophy*, $\vec{\Omega} = \frac{1}{2}\omega^2 dV$ (recall $\vec{\omega} = \nabla\times\mathbf{v}$).

• 3D MHD: total energy $\mathcal{E}$; magnetic helicity $\mathcal{H}_B = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{B}dV$; and *cross helicity*, $\mathcal{K} = \frac{1}{2}\int \mathbf{v}\cdot\mathbf{B}dV$

• 2D MHD: total energy $\mathcal{E}$; cross helicity $\mathcal{K}$; and the mean square magnetic potential, $\mathcal{A} = \frac{1}{2}\int \psi^2 dV$

2. CASCADE DIRECTIONS

In 3D HD turbulence we encountered a *forward cascade* (also called a *direct cascade*). That is, the turbulent shear stresses transfer energy to smaller scales (higher wavenumbers). The original Kolmogorov picture involved a forward cascade, of course, as does its modifications in the Kraichnan and GS models. Note that this allows a steady state spectrum $W(k)$: the forward cascade transfers energy from the input/driving scale, to the dissipation scale, at which scale it goes to heat. Because dissipation is more important at small scales (high k's), we expect always to be able to find a steady state.

For 3D-HD turbulence, we only have a forward cascade. But in 2D HD turbulence, and in 2D and 3D MHD turbulence, ond also finds an *inverse cascade*: power flows to *larger scales* (smaller wavenumbers). Inverse cascades move power to large scales, where dissipation is always small. Thus, they do not reach steady states. They tend to evolve to lower and lower k's, until they reach the scale of the system, and then accumulate power on those scales.

To summarize:

• 3D HD: all cascades are direct. This is the basic Kolmogorov picture.

• 2D HD: enstrophy has a forward cascade, but total energy has an inverse cascade. This seems to connect to the coherence and large-scale eddies seen in 2D turbulent flows.

• 3D MHD: energy and cross helicity have forward cascades; while helicity has an inverse cascade. This has dramatic consequences for maintaining large-scale ordered magnetic fields; this is the turbulent dynamo which we see in the next section.

• 2D MHD: energy and cross helicity have forward cascades; while the mean vector potential has an inverse cascade.

3. SELF-ORGANIZATION IN MHD

Some interesting properties of MHD turbulence are connected to these invariants.

• **Force-free fields**. We have already seen that turbulent relaxation evolves a plasma towards a self-organized state, in which $\mathbf{j} \parallel \mathbf{B}$. This is due to the invariance of the magnetic helicity (that is, to its much slower rate of decay than the magnetic energy shows).

• **Dynamic alignment**. There is evidence (in the solar wind, backed up by dynamical models) that strong MHD turbulence shows a second form of self-organization. It tends to evolve to a state in which $\pm\mathbf{v} \parallel \mathbf{B}$. That is, it shows *dynamic alignment* of the velocity and magnetic fields. This is due to the invariance of the cross-helicity; a formal minimization of $\mathcal{E}$, subject to $\mathcal{K}$ being constant, derives this condition.

• **Energy equipartition**. We should also note that there is evidence that MHD turbulence reaches a state of approximate equipartition between kinetic and magnetic energies: $\langle v \rangle^2 \sim \langle B^2 \rangle / 4\pi$. This is a common result in numerical simulations

D. MHD Dynamos

Now, a larger question: where do magnetic fields come from? In the lab, the answer is easy: "currents". In magnetic solids, the currents are those of well-ordered electrons spins in ferromagnetism. More typically, currents in the lab — and their consequent $\mathbf{B}$ fields — come from obvious things like batteries and wires. The issue is then, what drives the currents? My dictionary defines a dynamo as

> "a device for converting mechanical energy into electrical energy, usually by expending the mechanical energy in producing a periodic motion of a conductor and a surrounding magnetic field".

A simple lab version of this is called the *unipolar dynamo*, in Figure 24.1. This involves a conducting disk, threaded by a $\mathbf{B}$ field, which rotates about its axis. This induces a radial $\mathbf{E}$ field, $\mathbf{v} \times \mathbf{B}/c$, and thus a potential drop between the axis and the edge of the disk. If you hook up wires in the right way you'll have a current – and this current will create its own magnetic field.

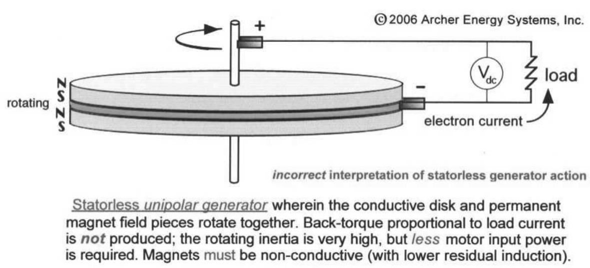

Figure 24.1. A simple unipolar dynamo (in a less than simple figure from www.stardrivedevice.com, the best figure I could find). The conducting disk moves through an (externally supported) $\mathbf{B}$ field as it rotates about its axis. The resultant EMF supports a potential drop between the axis and edge of the disk — which can drive a current.

What about astrophysical magnetic fields? To be specific, what is the origin of the earth's field, or the sun's field? It's easy to think of what doesn't work. One, even solid planets like the earth can't be ferromagnetic (because the core temperature is well above the Curie temperature at which permanent magnetism disappears); and clearly stars and galaxies can't be ferromagnetic at all. Two, we can't assume the fields are primordial — were somehow created when the sun/earth/galaxy formed — because we know the resistivity of the plasmas in question, and thus we know how long it would take a primordial current to dissipate. Such calculations predict that primordial fields would long ago have died away; but we know that stars, planets, and galaxies are still magnetized.[3]

Thus, we still must ask, "what supports astrophysical $\mathbf{B}$ fields? The answer is still, "currents"; but what drives astrophysical currents? We can't expect a device such as in Figure 24.1 exists inside a planet, or star, or whatnot ... so we need to find a way to drive *fluid motions* which can maintain the $\mathbf{B}$ fields we observe. This question gets us into what's called *dynamo theory*.

As with vorticity, the question of fluid dynamos could occupy a full course on its own. All we can do, once again, is a brief introduction. To set the stage, I paraphrase Roberts & Soward (1994, Ann Rev Astr Ap):

> We know dynamos exist; one can buy them in shops.[4] What, one may then ask, is all the fuss about? To answer, we must make an important distinction between commercial dynamos and naturally occuring ones. In order to ensure that the induced currents don't short-circuit, an engineer makes sure that the geometry of the machine is an asymmetric and multiply-connected region. By contrast, the conducting fluid in as astrophysica body usually occupies a symemtric, simply-connected domain – such as a sphere (the earth or sun) or a simple disk (the galaxy). It is by no means obvious that a dynamo can operate in such a simple system. Than, then, is the point of this field [by which they mean dynamo theory].

1. Cowling's Theorem

We can start by seeing what *won't* work. That is, most astrophysical models assumem simple, symmetric geometries; but these can't support a dynamo.

We can support the statement from Roberts & Soward by proving Cowling's theorem, namely, that *it is not possible to maintain a steady dynamo in an axisymmetric system*. To prove this, we follow Cowling's original (1934) argument, as presented by Choudhuri.

Start by assuming we do have an axisymmetric dynamo: one with $\partial/\partial t = \partial/\partial \phi = 0$. Consider a plane through the symmetry axis: the projections of the field

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[3] In addition, we know that the sun's field reverses pretty regularly, every 11 years or so; and the earth's field reverses less regularly, every $10^4 - 10^5$ years. This clearly requires some internal, self-governing mechanism.

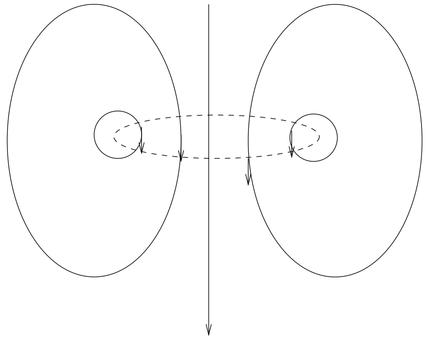

Figure 24.2. Illustration of geometry for Cowling's theorem. Following Choudhuri Figure 16.3.

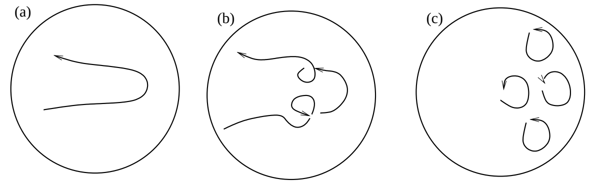

Figure 24.3. Parker's model of the solar dynamo, at the cartoon level. (a) Differential rotation (the sun's equator rotates faster than the poles) stretches initially dipolar field in the toroidal direction. (b) Coriolis forces acting on surface convective cells generates local poloidal fields. (c) The opposite sense of the Coriolis force in the north and south hemispheres, combined with the opposite sense of the initially toroidal field, results in a strong net poloidal field, rather than a randomly directed set of field loops. (These loops are shown projected in the meridional plane.) Following Choudhuri, Figures 16.4.

lines on this plane must be closed curves (think of a simple magnetic dipole). There will be at least one neutral point in this plane (a point where the closed field lines center) – and j_ϕ must be non-zero here, while $\mathbf{B}$ has only a ϕ component at this point. Take a line integral of Ohm's law (13.4) along a closed loop through these neutral points, enclosing the symmetry axis:

$$\frac{1}{\sigma}\oint j_\phi dl = \oint \mathbf{E}\cdot d\mathbf{l} + \oint \mathbf{v}\times\mathbf{B}\cdot d\mathbf{l} \qquad (24.7)$$

But now: the second term vanishes, because $\mathbf{B} \parallel d\mathbf{l}$ if this loop goes entirely through neutral points. The first term vanishes, because

$$\oint \mathbf{E}\cdot d\mathbf{l} = \int \nabla\times\mathbf{E}\cdot d\mathbf{S} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

But this last is zero by our steady-state assumption. However, the LHS of (24.7) is non-zero, as j_ϕ is finite. Thus, we have a contradiction; and Cowling's theorem is proved.

2. PARKER'S SOLAR DYNAMO

It follows, then, that we must relax the assumption of axisymmetry. The classic example of this is dynamo model Parker's (1955) model of the solar dynamo. His model is meant to describe the solar magnetic field, in particular explain how "turbulence" (in this case convection) offsets field stretching due to differential rotation, and support the quasi-dipolar field we observe.

His model is best presented qualitatively — refer to Figure 24.3 for the cartoon. Say the solar field starts mainly dipolar (this is roughly consistent with observations of the global field, just above the solar surface). The sun does not rotate as a solid body; near the surface, the equator rotates faster than the polar regions. This will stretch our dipolar field, generating toroidal components. Thus, it is no problem to generate toroidal field if the body has differential rotation. But this cannot be all of the story. Such a stretched toroidal field will have many local field-line reversals, and if nothing else happens it will simply decay away due to resistive dissipation.

However, the upper layers of the sun are convectively unstable. In this region, plasma blobs rise and fall.[5] Now, these vertically moving blobs are subject to a Coriolis force, due to the sun's overall rotation. The blobs therefore rotate as they rise; they act like little cyclones, and formally we say that their their motion has a net *helicity* (that means the small-scale motions do not have mirror symmetry: for instance a flow with $\mathbf{v}\cdot(\nabla\times\mathbf{v}) \neq 0$ is helical). Look at (b) of Figure 9.5: this cyclonic motion twists the magnetic field back into poloidal loops. Remember that both the direction of B_ϕ and of the Coriolis rotation are opposite in the north and south hemispheres: this means the direction of the poloidal field component generated is the *same* in the two hemispheres. We therefore have a fully working dynamo: poloidal fields are generated by the helical convective (turbulent) motions, while toroidal fields are generated by differential rotation. The whole system must be stabilized by dissipation – that is resistivity will keep each field component from getting too large.

Thus: we have argued "by cartoon" that helical, convective motions on the sun (or the earth) can maintain the large-scale $\mathbf{B}$ field. That is, we're arguing that *small-scale* turbulent motions can add up to a net *large-scale* dynamo, as long as the turbulence is helical. Now we need to quantify this idea.

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[5] In chapter 8 we talked about (in)stability to buoyancy – an unstable atmosphere will develop strong convection.

E. Kinematic Dynamos

One way to approach the problem is to assume you know the velocity field, and solve the induction equation in order to see what happens to $\mathbf{B}$. This type of problem is usually "illustrative" rather than "a solution"; I present one such example here (taken straight from Davidson).

We assume a *large-scale* (uniform, slowly varying in space) $\mathbf{B}_o$ field exists to start; and pick a useful *small-scale* velocity field,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_o e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{24.8}$$

where $\mathbf{v}_o = v_o(-i, 1, 0)$ in Cartesian (i.e., the v_x and v_y components of $\mathbf{v}$ are out of phase), and $\mathbf{k} \parallel \hat{\mathbf{z}}$ (so the wave travels along the z-axis). This is a helical wave: $\nabla \times \mathbf{v} = k\mathbf{v}$. What effect does this have on the magnetic field? We can linearize: assume $\mathbf{B} = \mathbf{B}_o + \mathbf{b}$, where $\mathbf{B}_o$ is some background field. Because we have two very different spatial scales, we can split the linearized induction equation into "large-scale" and "small-scale" parts. The large-scale part is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\langle \mathbf{v} \times \mathbf{b} \rangle) + \eta \nabla^2 \mathbf{B} \tag{24.9}$$

where the first term on the RHS – the inductive term – derives from the *mean* (over space & time) of the small-scale EMF. The small-scale part is

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_o) + \eta \nabla^2 \mathbf{b} \tag{24.10}$$

Note, in (24.9) we've made a critical assumption: that the mean EMF from the small-scale stuff, $\langle \mathbf{v} \times \mathbf{b} \rangle$, has large-scale order.

To solve this, we also assume $\mathbf{b} \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ (it echoes the wave behavior). If we carry out the algebra, we can solve for the amplitude and direction of $\mathbf{b}$:

$$\mathbf{b} = \mathbf{b}\frac{(\mathbf{B}_o \cdot \mathbf{k})(-\omega + 1\eta k^2)}{(\eta^2 k^4 + \omega^2)}\hat{\mathbf{z}} \tag{24.11}$$

From this, we can (eventually) show that the helical $\mathbf{v}$ field, and the $\mathbf{b}$ field that responds to it, do have a finite, time-averaged EMF:

$$\langle \mathbf{v} \times \mathbf{b} \rangle = (\mathbf{B}_o \cdot \mathbf{k})\frac{\eta k^2 v_o^2}{(\eta^2 k^4 + \omega^2)}\hat{\mathbf{z}} \tag{24.12}$$

This verifies our assumption, that $\langle \mathbf{v} \times \mathbf{b} \rangle$ belongs in the large-scale induction equation, (24.9). Thus, this EMF acts back on the large-scale field, in fact acts as a source (driving) term on that field.

F. Mean-Field Dynamos

The above is one specific example of a "turbulent helical dynamo". The problem is usually treated much more formally, using statistical measures of MHD turbulence.

This analysis – backed up by numerical work – shows that MHD turbulence can, indeed, maintain magnetic fields with structure on scales large compared to the turbulent scales. This result derives from the fact that magnetic helicity cascades to larger, rather than smaller, scales.

We return to mean-field theory to describe this. Split the velocity and magnetic fields again into mean and fluctuating parts:

$$\mathbf{B} = \mathbf{B} + \mathbf{b}\,; \qquad \mathbf{v} = \mathbf{V} + \mathbf{v} \tag{24.13}$$

and assume that $\mathbf{b}$ and $\mathbf{v}$ have zero mean. Once again, we split the induction equation into large-scale (mean) and small-scale (fluctuating) parts. For the mean field, we get

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}_o) - \nabla \times \boldsymbol{\varepsilon} + \eta \nabla^2 \mathbf{B} \tag{24.14}$$

where

$$\boldsymbol{\varepsilon} = -\langle \mathbf{v} \times \mathbf{b} \rangle \tag{24.15}$$

is the net EMF due to the fluctuating $\mathbf{v}$ and $\tilde{\mathbf{B}}$ fields. Once again, we're assuming that $\boldsymbol{\varepsilon}$ belongs in the large-scale equation – but we need to verify that.

So, we must ask whether $\boldsymbol{\varepsilon}$ has any interesting large-scale effect. Because we're being general (not specific as in the previous example), we need to be formal here. If $\mathbf{v}$ and $\mathbf{b}$ are rapidly varying, have zero mean, and uncorrelated, we'd expect the mean of their product to be zero. It turns out ("can be shown") that things are interesting (*i.e.*, $\boldsymbol{\varepsilon} \neq 0$) if the turbulence satisfies two conditions: (i) it must be helical, satisfying $\mathbf{v} \cdot \nabla \times \mathbf{b} \neq 0$; and (ii) it must be resistive; $\eta \neq 0$. If both of these conditions are met, one can show formally that the turbulent EMF, $\boldsymbol{\varepsilon}$, will be non-zero *on large scales*.

But we also want to explore the small-scale behavior. The small-scale field satisfies

$$\begin{aligned}\frac{\partial \mathbf{b}}{\partial t} =& \nabla \times (\mathbf{V} \times \mathbf{b} \mathbf{v} \times \mathbf{B}) \\ &+ \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b}\end{aligned} \tag{24.16}$$

where

$$\mathbf{G} = \mathbf{v} \times \mathbf{b} - \langle \mathbf{v} \times \mathbf{b} \rangle \tag{24.17}$$

Now: the standard analysis argues that the $\nabla \times \mathbf{G}$ term can be dropped from (24.16) (due to being second-order small), as can the dissipative term $\eta\nabla^2\mathbf{b}$ (assuming we are still far from the disspation lengths). This means that we can formally integrate (24.16) to get

$$\mathbf{b} \simeq \int \nabla \times [\mathbf{v} \times \mathbf{B}]\, dt' \tag{24.18}$$

(where $\mathbf{v}'$ is evaluated at t', and unprimed means evaluated at t. From this,

$$\boldsymbol{\varepsilon} \simeq -\int dt' \langle \mathbf{v} \times \big[\nabla \times \big(\mathbf{v}' \times \mathbf{B}\big)\big]\rangle \tag{24.19}$$

But now, we can note that $\boldsymbol{\varepsilon}$ is linear in $\mathbf{B}$ and its curl:

$$\boldsymbol{\varepsilon} \simeq \alpha\mathbf{B} + \beta\nabla \times \mathbf{B} \tag{24.20}$$

Working out the integrals in detail, one finds

$$\begin{aligned} \alpha &= \frac{1}{3}\int dt' \langle \mathbf{v}\cdot\nabla\times\mathbf{v}'\rangle = \frac{\tau}{3}\mathcal{H}_V \\ \beta &= \frac{1}{3}\int dt' \langle \mathbf{v}\cdot\mathbf{v}'\rangle = \frac{\tau}{3}\langle \mathbf{v}^2\rangle \end{aligned} \tag{24.21}$$

if τ is the velocity correlation time. Thus: the mean field equation now becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}_o) + \nabla \times (\alpha\mathbf{B}) + (\eta + \beta)\nabla^2\mathbf{B} \tag{24.22}$$

Thus: the β term simply adds to the dissipation – another meeting with turbulent dissipation. The α term, however, acts as a source term: one can show that it leads to a growth of magnetic flux.

Thus, we have a big result: helical turbulence in a plasma can amplify the large-scale field. That's a *turbulent dynamo*.

If this works – if $\boldsymbol{\varepsilon} = \alpha\mathbf{B}$ — then we can see two useful astrophysical consequences. One is balancing ohmic losses, as in the sun or the earth — and (in principle) accounting for the occasional field reversals in each body. The second is "growing" the $\mathbf{B}$ field in the first place. To see this, note that (9.14) allows solutions $\mathbf{B} \propto e^{\alpha t}$, if α is constant in time. That's a growing $\mathbf{B}$ field, with growth time $\sim L/\alpha$ (some large-scale length scale L). We might expect that a small seed field would grow exponentially until some other physics (dissipation? back reaction on the driving fluid?) comes into play.

In addition, we note that when $\mathbf{V} = 0$ (no large-scale flows), (24.14) has particularly simple steady-state solutions: $\mathbf{B} \parallel \mathbf{j}$. That is just a force-free field – which connects back to Taylor relaxation, which we saw in Chapter 15: a plasma will spontaneously relax to a mininum-energy state, which is force-free. The relaxation proceeds through self-generated turbulence – thus the plasma finds its own solution of $\nabla \times (\alpha\mathbf{B}) + \eta\nabla^2\mathbf{B} = 0$.

G. Astrophysical dynamos in the lab

Finally, a few words about trying to do this in the lab. Everything above is still pure theory — it would be good to verify directly that an $\alpha\omega$ dynamo (rotation plus turbulence, as in the sun), or an α^2 dynamo (pure turbulence) can really make a large-scale ordered $\mathbf{B}$ field. Several groups are working on this, including NMT's very own Stirling Colgate. The experiments use liquid metal — usually liquid sodium — in some sort of rotating system (the ω in an $\alpha\omega$ dynamo), and try various ways to induce turbulence (the α) in the flow. The last I heard, no one had sucessfully made their dynamo work — but I think the field's progressing. Check the Feb 2006 issue of *Physics Today* if you'd like more details.

References

For the IK/GS turbulent cascade discussion, I've followed two excellent recent review papers: Diamond etal, *A Tutorial on basic concepts in MHD turbulence and turbulent processes* (available through the physics.ucsd web site, I can't find the publication reference); and Schekochihin & Cowley, *Turbulence and magnetic fields in astrophysical plasmas*, to appear in *MHD: historical evolution and trends* (Molokov, Moreau & Moffatt, eds, 2006). These papers have references to the original papers.

I've also used Biskamp's book for discussions on intermittency, inverse cascades & invariants, etc.

The general dynamo material is in lots of books; I like Priest's *Solar MHD*, and also Moffett's original *Mean Field Electrodynamics*.