3. POTENTIAL FLOW

In the first chapter we wrote down fairly general (partial differential) equations describing fluid flow. We now must look for solutions. In this chapter we consider one of the traditional methods, *potential flow*, which describes incompressible flows in which the velocity field has no curl.

A. Setups and Assumptions

Our first set of solutions consider flows with no vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0$. In addition, we will generally assume the flows are incompressible. Why are such restrictive solutions interesting? First, we can find analytic solutions: so let's see where they lead. Second, these solutions have some relation to reality . . . because vorticity turns out to be a (nearly) conserved quantity, and hard to generate away from boundaries: we will prove this later. Thus, flows which start out irrotational tend to stay this way.

1. VELOCITY POTENTIAL

For irrotational flows, the fact that $\nabla \times \mathbf{v} = 0$ allows the velocity to be written as the gradient of a scalar potential:

$$\mathbf{v} = \nabla\phi \tag{3.1}$$

Explicitly, this is

$$v_x = \frac{\partial\phi}{\partial x}\ ; \qquad v_y = \frac{\partial\phi}{\partial y} \tag{3.2}$$

Further, when the flow is incompressible – which is the case in all examples we will see — we have $\nabla \cdot \mathbf{v} = 0$, so that

$$\nabla^2\phi = 0 \tag{3.3}$$

That is, the velocity potential satisfies Laplace's equation. But this is great if we are traditional analytic physicists: we can use all the (glorious) solution methods worked out in electrostatics. This will be our general approach – picking examples where we can find solutions to Laplace's equation, based on what instinct we have from analytic E&M. Once we have found $\phi(x, y)$, the velocity solution comes immediately from (3.1).

2. STREAM FUNCTION

If the flow is two-dimensional, we can also define another useful function. This is the stream function, ψ. Define that

$$\mathbf{v} = -\nabla \times (\psi\hat{\mathbf{z}}) \tag{3.4}$$

which becomes, explicitly,

$$v_x = \frac{\partial\psi}{\partial y}\ ; \qquad v_y = -\frac{\partial\psi}{\partial x} \tag{3.5}$$

(Why is this possible? From 3.4) it's clear that $\nabla \cdot \mathbf{v} = 0$. Thus, this works for incompressible flows.) The stream function has several useful properties.

- It's related to the velocity potential; we can get one from the other:

$$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}\ ; \qquad v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{3.6}$$

- ψ also satisfies Laplace's equation; we can solve for ψ rather than ϕ if we wish.
- Streamlines are tangent to the local velocity $\mathbf{v}$. A streamline obeys

$$v_y dx = v_x dy \tag{3.7}$$

(compare 1.1). Thus, the differential of ψ along some path is

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \tag{3.8}$$

Combining (3.7) and (3.8) shows that $d\psi = 0$ along a streamline. We can therefore use the function ψ to label streamlines – as in the examples.
- Finally, using (3.6), it follows directly that $\nabla\phi \cdot \nabla\psi = 0$, so that streamlines and equipotential lines cross each other at right angles. This again says that $\mathbf{v}$ is tangential to the contours of constant ψ – so the value of ψ provides a label for the streamlines in the flow.

3. OTHER COORDINATES

As long as the problem is two dimensional and incompressible, we can define ϕ and ψ functions.

Plane Polar Coordinates

The velocity is connected to the potential by

$$v_r = \frac{\partial\phi}{\partial r}\ ; \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{3.9}$$

The incompressibility condition is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = 0 \tag{3.10}$$

From this we can define a stream function:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\ ; \qquad v_\theta = -\frac{\partial\psi}{\partial r} \tag{3.11}$$

Cylindrical Coordinates, Axisymmetric

The velocity is connected to the potential by

$$v_r = \frac{\partial \phi}{\partial r} \; ; \qquad v_z = \frac{\partial \phi}{\partial z} \tag{3.12}$$

The incompressibility condition is

$$\frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right) = 0 \tag{3.13}$$

From this we can define a stream functionn:

$$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \; ; \qquad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{3.14}$$

Spherical Polar Coordinates, Axisymmetric

The velocity is connected to the potential by

$$v_r = \frac{\partial \phi}{\partial r} \; ; \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{3.15}$$

The incompressibility condition is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) = 0 \tag{3.16}$$

From this we get a stream function:

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \; ; \qquad v_\theta = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r} \tag{3.17}$$

B. Two-dimensional Planar Problems

Consider Laplace's equation in two dimensions, in plane polar coordinates (r, θ).

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

We know the possible general solutions:

$$\phi = \text{constant} \; ; \quad \phi \propto \ln r \; ; \quad \phi \propto r^n \cos(n\theta) \tag{3.18}$$

And, of course, any superposition of these is a solution. So, we proceed by finding useful superpositions . . .

One important building block will be a simple source or sink:

$$\begin{aligned} \psi &= \frac{C\theta}{2\pi} \; ; \quad \phi = \frac{C}{2\pi}\ln r \\ v_r &= \frac{C}{2\pi r} \; ; \quad v_\theta = 0 \end{aligned} \tag{3.19}$$

This describes radial flow away from a source (or inwards to a sink if $C < 0$).

Another basic building block is a steady flow at U is represented in Cartesian by

$$\phi = Ux \; ; \qquad \mathbf{v} = \nabla\phi = U\hat{\mathbf{x}} \tag{3.20}$$

1. SOURCES AND SINKS, FLOW PAST HALF BODY

We can combine these two building blocks (and work in cylindrical):

$$\begin{aligned} \psi &= Ur\sin\theta \\ \phi &= Ur\cos\theta + \frac{C}{2\pi}\ln r \\ v_r &= U\cos\theta + \frac{C}{2\pi r} \\ v_\theta &= -U\sin\theta \end{aligned} \tag{3.21}$$

to get what is called flow past a half body. We see why, by noting that the corresponding stream function is

$$\psi = Ur\sin\theta + \frac{C}{2\pi}\theta \tag{3.22}$$

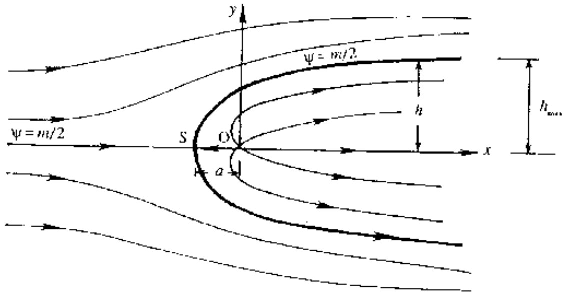

Figure 3.1. Potential flow past a two-dimensional "half body" (represented mathematically by a source function). The boundary streamline is labelled by $\psi = m/2$ (that's $C/2$ in our notation); the shape of this streamline defines the shape of the half body. S shows the stagnation point in front. From Kundu figure 6.9

The flow clearly has a stagnation point: if it's at position (a, π), then $\mathbf{v}(a, \pi) = 0$ requires $a = C/2\pi U$. Now, the value of the stream function passing through this point – its label – is

$$\psi_s = Ua\sin\pi + \frac{C}{2\pi}\pi = \frac{C}{2}$$

and the equation of the streamline passing through this point is found by taking $\psi(r, \theta) = \psi_s = C/2$:

$$Ur\sin\theta + \frac{C}{2\pi}\theta = \frac{C}{2} \tag{3.23}$$

This stagnation streamline separates the flow into an upstream region ($\mathbf{v} \to U\hat{\mathbf{x}}$ at ∞), and the region inside the streamline, a region with a smooth nose, called a *half body*. We can, therfore, put a physical body with this shape into a flow, and keep the same external solution.

Once we have the velocity solution, we can find the drag on the half body – the net force the fluid exerts on the body (or vice versa). As long as we ignore friction (*i.e.*, viscosity), the net force on the body is just the (vector) integral of the pressure over the surface of the body. That is, for this simple case, we expect the drag force to be

$$\mathbf{F}_D = -\int_S p\hat{\mathbf{n}}dS \tag{3.24}$$

if $\hat{\mathbf{n}}$ is the normal to the surface S, and the integral is taken over the surface of the body. Bernoulli‘s relation for an incompressible fluid (1.19) can be written as

$$p + \frac{1}{2}\rho v^2 = p_\infty + \frac{1}{2}\rho U^2 \tag{3.25}$$

where we have evaluated the constant at “infinity”, that is far upstream. Work this out: the (normalized) excess pressure,

$$C_p = 2(p - p_\infty)/\rho U^2$$

turns out to be positive at the front of the body, zero at $\theta = 113°$, and negative along the sides. You can show from this, by setting up the integrals, that the net pressure, integrated over the surface of the body, is zero: there is no drag in this system.

2. FLOW PAST A CYLINDER

In another example, we start with a *doublet*: a paired source and sink, of equal strength C, separated by ϵ; let them approach each other while the product of their separation and their strength stays finite, $C\epsilon/\pi \rightarrow \mu$. The potential and stream function, from Kundu, are

$$\phi = \frac{\mu x}{x^2+y^2}\ ; \qquad \psi = -\frac{\mu y}{x^2+y^2} \tag{3.26}$$

The streamlines, given by $\psi = constant$, are circles whose centers lie on the y-axis, and which are tangent to the x-axis at the origin.

Now: combine this with a uniform stream flow, $\phi = Ux$ and $\psi = Uy$: combining and putting in polars, with $a^2 = \mu/U$, gives

$$\begin{aligned} \phi &= U\left(r + \frac{a^2}{r}\right)\cos\theta\ ; \quad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta \\ v_r &= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\ ; \quad v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \end{aligned} \tag{3.27}$$

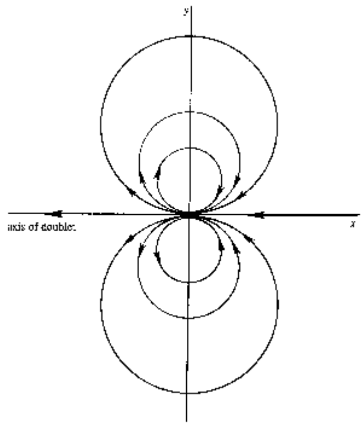

Figure 3.2. A doublet: a source and a sink which approach each other, keeping the product of their strength C and separation ϵ constant. The streamlines become circles, whose centers lie on the y-axis and which are tangent to the x-axis at the origin. From Kundu figure 6.8

But, now, this shows that $\psi(a,\theta) = 0$: the streamline $\psi = 0$ is a circular cylinder of radius a. Thus, flows inside this cylinder have no influence on flows outside; we can again replace this doublet with a physical cylinder, and find the same flow outside.

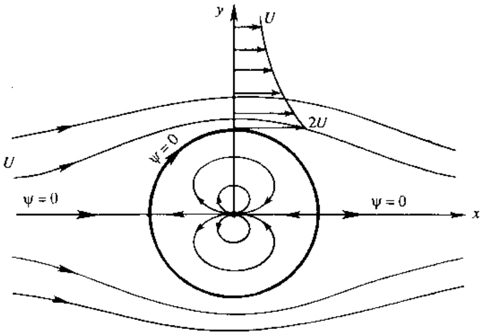

Figure 3.3. Potential flow past a circular cylinder, described by the $\psi = 0$ streamline. The inside of the cylinder is represented mathematically by a doublet. From Kundu figure 6.11

Finally, we can again work out the drag. The pressure difference

$$C_p = \frac{2(p-p_\infty)}{\rho U^2} = 1 - \frac{v^2}{U^2} = 1 - 4\sin^2\theta \tag{3.28}$$

so, again, is positive at the front and negative at the back . . . so that, once again, there is no net pressure drag on this cylinder.

C. Axisymmetric 3D Problems

Axisymmetric 3D problems are also amenable to this type of solution. Spherical polar coordinates are appropriate here. The general solutions to Laplace's equations for this geometry are

$$\phi = \text{constant} \; ; \quad \phi \propto r^n P_n(\cos\theta) \; ; \\ \phi \propto r^{-(n+1)} P_n(\cos\theta) \tag{3.29}$$

which are just your favorite (axisymmetric) Laplace's solutions in spherical polars. The velocity potentials and stream functions obey

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \; ; \\ v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{3.30}$$

We can run through some familiar-looking solutions.

POINT SOURCE AND STREAM FLOW

These are our two most basic. A point source has the potential and stream function,

$$\phi = -\frac{Q}{4\pi r} \; ; \qquad \psi = -\frac{Q}{4\pi}\cos\theta \tag{3.31}$$

where Q (units: m^3/s) is the discharge rate of the source. A uniform stream flow has the potential and stream function,

$$\phi = Ur\cos\theta \; ; \qquad \psi = \frac{1}{2}Ur^2\sin^2\theta \tag{3.32}$$

The velocity solutions for these two basics are straightforward. It is worth noting that the sum of these two gives the velocity field for an axisymmetric half body,

$$v_r = U\cos\theta + \frac{Q}{4\pi r^2} \; ; \qquad v_\theta = -U\sin\theta \tag{3.33}$$

FLOW AROUND A SPHERE

As with flow past a cylinder, this adds a uniform stream to a doublet which opposes the stream.

A Doublet, in this geometry, has the potential and stream function,

$$\phi = \frac{m}{r^2}\cos\theta \; ; \qquad \psi = -\frac{m}{r}\sin^2\theta \tag{3.34}$$

The velocity solutions for flow around a sphere are straightforward. We have, then

$$\phi = Ur\cos\theta + \frac{m}{r^2}\cos\theta \; ; \\ \psi = \frac{1}{2}Ur^2\sin^2\theta - \frac{m}{r}\sin^2\theta \tag{3.35}$$

From this, we see that $\psi = 0$ for $\theta = 0, \pi$ at any r (that is, the x-axis), and also for $r = a = (2m/U)^{1/3}$ (any θ). Thus, again, we have a spherical surface, flows inside of which have no effect on the outer stream flow.

Again, then, we can work out the pressure/drag on this system. The velocity components are

$$v_r = U\left[1 - \left(\frac{a}{r}\right)^3\right]\cos\theta \; ; \\ v_\theta = -U\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\sin\theta \tag{3.36}$$

so that the pressure is

$$p(a,\theta) = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right) \tag{3.37}$$

So, once again, this is symmetric in θ, leading to zero drag on the sphere. Finally, let me refer you to Faber for a good discussion of real-world effects in this problem, §4.7.

A LINE SOURCE

describes a mass source which is distributed in a line; let it have a source function k per length, so that its total source is $Q = \int k dx$.

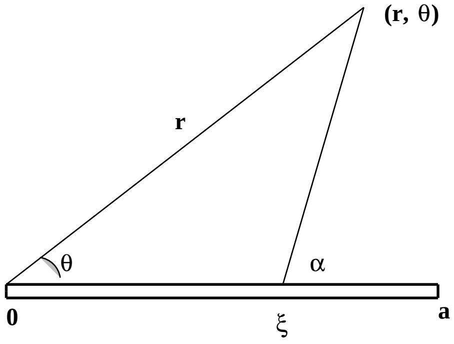

Figure 3.4. The geometry of a line source, which extends from $\xi = 0$ to $\xi = a$.

Each differential bit of the line source has potential and stream functions,

$$d\phi = -\frac{kd\xi}{4\pi r} \; ; \qquad d\psi = -\frac{kd\xi}{4\pi}\cos\alpha \tag{3.38}$$

if ξ is the running variable along the length of the source. We also let $R = r\sin\theta$ be the vertical distance to the observation point, and $r_1 = \left(r^2 + a^2 - 2ra\cos\theta\right)^{1/2}$ be the radial distance from $\xi = a$ (the end of the source) to the observation point. The net stream function is then

$$\psi(r,\theta) = \int_0^a d\psi = -\frac{k}{4\pi}\int_0^a \cos\alpha(\xi)d\xi = \frac{k}{4\pi}(r - r_1) \tag{3.39}$$

where the last step is obtained by straightforward geometrical identities and changes of the integration variable. [Exercise: what is $\phi = \int d\phi$?]

D. d'Alembert's "Paradox"

We have now seen several examples of flow past a solid body, described by potential flow. In each case we found that the net force exerted by the flow on the body is zero. Thus, there is no drag in a potential flow around a body. This result turns out to hold for any body, symmetric or not.

Does this sound wrong? We know experimentally that a drag force does exist when a body is immersed in a fluid flow. This apparent contradiction, between the theory and experiment, became known as d'Alembert's paradox. As you might suspect, the solution of the paradox is that the theory is incomplete (no, we don't say "wrong"). In particular, we have ignored two important pieces of the physics here. One, is that these solutions can have nonzero velocity at a solid surface: check Figure 3.1 or 3. This is naive: any real fluid has a finite, if small, viscosity. But viscosity will force to a no-slip boundary condition. This leads to a viscous boundary layer, which will modify the solutions close to the surface, and add tangential stress which contributes some drag. Two, the flow behind the body may very well be turbulent – as this viscous boundary separates from the body and conects into the wake. We will see later that this also contributes to a drag force. Both of these ideas are illustrated in Figure 3.6.

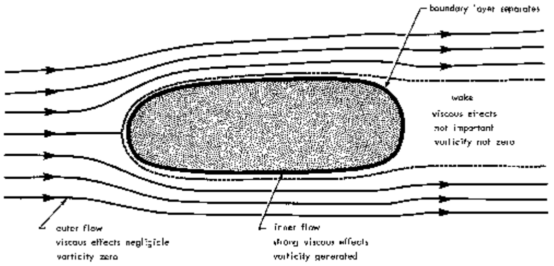

Figure 3.5. Illustrating the nature of the flow field around an arbitrarily shaped body. From Currie Figure 9.1.

We can also carry out a general drag calculation, for a finite body, that illustrates this argument. We could work out the pressure in the fluid, everywhere on the surface of the body, and then integrate it over the surface. Or we can try a different approach.

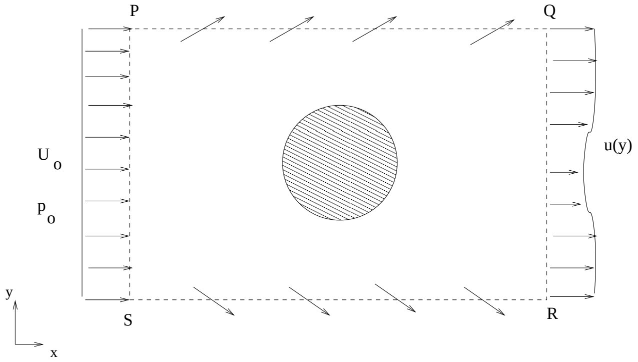

Figure 3.6. Momenteum balance of flow past an arbitrary body. The upstream flow has constant speed U_o and pressure p_o. The downstream velocity may be a function of y; in this sketch it is lower immediately behind the body (as due to a wake). Note, the body need not be spherical, that was just easy for me to draw... Following Kundu figure 4.7.

Consider the region of fluid within the rectangle SRQP; the net drag on the body within this region is just the rate of change of momenteum flux: $F_D = \dot{P}_{out}$, *i.e.* the net outflow of x-momentum from the region. The outflow through PS and QR is easy to calculate:

$$\begin{aligned} \dot{P}^{PS} &= -\int_o^L U_o \rho U_o dy = -L\rho U_o^2 \ ; \\ \dot{P}^{QR} &= \rho \int_0^L u(y)^2 dy \end{aligned} \tag{3.40}$$

Now: there must be mass outflow, $\dot{M}$, through the sides, SR and PQ (this is necessary to conserve mass if $u(y) < U_o$, as shown). Mass conservation requires

$$L\rho U_o = \dot{M}^{PQ} + \dot{M}^{SR} + \rho \int_O^L u(y) dy \tag{3.41}$$

But if the sides are far enough away from the rock that the x velocity $\simeq U_o$ there, then the rate of outflow of x-momentum is

$$\dot{P}^{PQ} + \dot{P}^{SR} = \rho U_o \int_0^L (U_o - u(y)) dy \tag{3.42}$$

Finally, we can collect all of this, to find the net drag on the body: $F_D = \dot{P}^{out} = \dot{P}^{PS} + \dot{P}^{QR} + \dot{P}^{PQ} + \dot{P}^{SR}$. This gives,

$$F_D = \dot{P}^{out} = \rho \int_O^L u(y)(U_o - u(y)) dy \tag{3.43}$$

And .. referring back to our simple example of potential flow past a cylinder, note from (3.27) that the downstream velocity field $\rightarrow U$ as $r \rightarrow \infty$. That is, there is no slow-down of the flow in this idealized case; and thus no drag. Drag comes from viscosity and boundary effects which generate a wake behind the body.

References: I've mostly followed Kundu here.