4. VORTICITY

For God's Sake Let us sit upon the ground
and tell sad stories
Of vortex filaments.
How some have been ill-posed, some singular,
Some poisoned by their self induction, some core size killed,
Some haunted by the mathematics they have involved.

All murderous.

For within the swirling motion that rounds the mortal circulation
Of a vortex
Keeps futility his court,
And there the non-linearity sits
Scoffing at his state and grinning at his theories
Allowing him a breath, a little scene to linearize, compute and fill with approximations
And then at last he comes and with a little inconsistency bores through the costly hopes and

Farewell . . .

Shakespeare, *Richard II*, Act 3, Scene 2
"translated" by H. C. Yuen, quoted in
Vortex Dynamics, P. G. Saffman

A. Vortex Kinematics

Vorticity is a large topic, and we can only treat it lightly here. We start with kinematics. We saw an example in Chapter 3 of a flow field with a net circulation: in which the line integral of the velocity, around a closed loop, is finite. Such a flow has a net circulation. We generalize this here, with differential and integral definitions.

Differential: *vorticity* is the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{4.1}$$

We can think of **vortex lines** – just like magnetic field lines or streamlines, they are lines which are everywhere tangent to the local vorticity. Also we will run into **vortex tubes**, regions of high vorticity (picture a tornado; that is an extreme example).

Integral: the magnitude of the *circulation* is the vorticity integrated around a closed path:

$$K = \oint \mathbf{v} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{v}) \cdot d\mathbf{A} \tag{4.2}$$

where Stokes' theorem is used in the second equality. As we noted above, $\mathbf{K}$ can be taken as a vector, with direction given by the sense of the flow, using the right hand rule. We can think of K as a "flux" of vorticity; it measures the amount of $\boldsymbol{\omega}$ passing through some surface bounded by $d\mathbf{l}$.

Let's look at some examples. *Solid body rotation* is described by $v_\theta = \Omega r$;

$$\begin{aligned} \omega &= \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = 2\Omega \; ; \\ K &= \int_0^{2\pi} v_\theta r d\theta = 2\pi r^2 \Omega \end{aligned} \tag{4.3}$$

so solid body rotation has a net vorticity, and a net circulation.

An Irrotational vortex comes from potential flow. Pick $v_\theta = C/r$; then $r v_\theta =$ constant, so that

$$\begin{aligned} \omega &= 0, r \neq 0 \; ; \\ K &= \int_0^{2\pi} v_\theta r d\theta = 2\pi C \end{aligned} \tag{4.4}$$

Thus, circular streamlines do not imply that the flow has a net vorticity everywhere. In this example $\omega \neq 0$ only at the origin; elsewhere the flow is curl-free. The divergeance at the origin, however, is enough to give this vortex a net circulation.

The Rankine vortex is a more physical extension of the irrotational one. A real vortex must have a rotational core (we can't have $v_\theta \to \infty$). We can approximate this with $v_\theta \propto r$, $r \leq R$ (the radius of the *core* of the vortex); and $v_\theta \propto 1/r$ for $r > R$. Thus, the Rankine vortex has uniform vorticity in its core, and zero elsewhere.

B. Vortex Dynamics

We will only need this result in the nearly-incompressible limit. That is, we take $\nabla \cdot \mathbf{v} \simeq 0$, but formally allow $\nabla \rho \neq 0$. We start with the incompressible form of the Navier-Stokes equation, from (2.38),[1]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = -\nabla\Phi - \nu \nabla \times (\nabla \times \mathbf{v}) \tag{4.5}$$

[1] The viscous term is written slightly differently, by the magic of vector analysis.

To proceed, take the curl of (4.5). Using the vector relation $\mathbf{v}\cdot\nabla\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{v} + \frac{1}{2}\nabla v^2$, we get the *dynamical equation for vorticity*:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) = \nu\nabla^2\boldsymbol{\omega} + \frac{1}{\rho^2}\nabla\rho\times\nabla p \quad (4.6)$$

or[2]

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{v} + \nu\nabla^2\boldsymbol{\omega} + \frac{1}{\rho^2}\nabla\rho\times\nabla p \quad (4.7)$$

A possible further step is to consider the *barotropic limit*, in which the pressure is only a function of the density [$p = p(\rho)$], for instance in an adiabatic gas with p/ρ^γ = constant. In that case, $\nabla\rho \parallel \nabla p$, so their cross product is zero and the equations simplify, to

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) = \nu\nabla^2\boldsymbol{\omega} \quad (4.8)$$

and

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{v} + \nu\nabla^2\boldsymbol{\omega} \quad (4.9)$$

Yet another possible step is to work in a rotating frame. Referring back to chapter 1, the same analysis gives here,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\mathbf{v} + \nu\nabla^2\boldsymbol{\omega} + \frac{1}{\rho^2}\nabla\rho\times\nabla p \quad (4.10)$$

C. Conservation of Circulation: Kelvin's Theorem

A basic result here is **Kelvin's theorem**: *in an inviscid, barotropic flow with conservative body forces, the circulation K around a closed curve, moving with the fluid, is a constant of the motion.* This means that vortex lines can be thought of as "frozen into" the fluid: if you stretch or compress a piece of fluid, any vortex tube running through it must also be stretched or compresses.

This is worth two slightly different proofs, as follows.

[2] If we want to eliminate the nearly incompressible assumption, we simply need to start with (2.37) and carry out the full vector curl, to go from $\mathbf{v}$ to $\boldsymbol{\omega}$; students with time on their hands are welcome to do this.

Proof #1 Kundu proves this as follows. Start with Euler's equation, (1.10) (note we aren't worried about viscosity right now):

$$\frac{D\mathbf{v}}{Dt} = \frac{1}{\rho}\nabla p + \nabla\Phi \quad (4.11)$$

Now, take the formal derivative of K:

$$\frac{DK}{Dt} = \oint \frac{D\mathbf{v}}{Dt}\cdot d\mathbf{l} + \oint \mathbf{v}\cdot\frac{D}{Dt}d\mathbf{l} \quad (4.12)$$

But using Euler's equation, and noting that $\nabla p\cdot d\mathbf{l} = dp$, the difference in pressure between two adjacent points:

$$\frac{DK}{Dt} = \oint \nabla\Phi\cdot d\mathbf{l} - \oint\frac{dp}{\rho} + \oint \mathbf{v}\cdot\frac{D}{Dt}d\mathbf{l} \quad (4.13)$$

Now: the force, being from a potential, is conservative; so the first term goes to zero. If the density is only a function of pressure, $\rho = \rho(p)$ – that is if the fluid is barotropic – then dp/ρ is a perfect differential, so that the second integral (around a closed curve) also goes to zero. Finally, the third term involves $D(d\mathbf{l})/Dt = d\mathbf{v}$, and thus $\mathbf{v}\cdot D(d\mathbf{l})/Dt = D(v^2/2)/Dt$, also a perfect differential, so the third term also vanishes. Thus, K is a constant of the motion in this flow.

Proof #2 I have not seen this in a text, but it follows directly from magnetic field flux freezing arguments (which will be presented in a later chapter). Here, we keep the viscosity term around, but retain the incompressible, barotropic limit. That means we are again using (4.5) as the governing equation, and (4.6) derived from it, but now assuming $\nabla\rho\times\nabla p = 0$. The formal derivative of K can now be written

$$\frac{\partial K}{\partial t} = \int_A \left[\frac{\partial\boldsymbol{\omega}}{\partial t} - \nabla\times(\mathbf{v}\times\boldsymbol{\omega})\right]\cdot d\mathbf{A} = \int \nu\nabla^2\boldsymbol{\omega}\cdot d\mathbf{A} \quad (4.14)$$

In expanding out the K integral, we have allowed for intrinsic changes of $\boldsymbol{\omega}$ with time, and also for motion of the boundary of the surface, $\mathbf{l}$. But now the result is clear: if $\nu\to 0$, $K\to$ constant. Furthermore, a finite viscosity will lead to eventual dissipation of the circulation.

The **Helmholtz theorems** are related; they can be derived from the same assumptions that we used for Kelvin' theorem. The theorems are:

- Vortex lines move with the fluid.
- The strength of a vortex tube – the circulation – is constant along its length.

• A vortex tube cannot end within the fluid. It must either end at a solid boundary, or form a closed loop (a *vortex ring*).

• The strength of a vortex tube is constant in time.

D. The Magnus Effect

Let's return to potential flow. Consider, again, flow past a cylinder, but this time let the cylinder rotate. This situation leads to a net force on the cylinder.

To be specific, consider a potential, and stream function, given by

$$\phi = \frac{K}{2\pi}\theta\,; \qquad \psi = -\frac{K}{2\pi}\ln r \tag{4.15}$$

which recovers the velocities $v_r = 0$; $v_\theta = K/2\pi r$ and streamlines $r =$ constant, as we expect. (Note, the r term in the log in the expression for ψ must contain a scaling parameter to make it dimensionless; this will not matter as we only take derivatives of ψ). Thus, this potential-stream pair describes *counterclockwise* flow (if $K > 0$) about the origin. Or, put in a minus sign to make *clockwise* flow, and add this to our previous solution for irrotational flow around a cylinder:

$$\begin{aligned} \phi &= U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{K}{2\pi}\theta\,; \\ \psi &= U\left(r - \frac{a^2}{r}\right)\sin\theta - \frac{K}{2\pi}\ln r \end{aligned} \tag{4.16}$$

We recall a is a length scale, corresponding to the radius of the cylinder. This potential/stream pair has the velocity field,

$$\begin{aligned} v_r &= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \\ v_\theta &= -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{2\pi r} \end{aligned} \tag{4.17}$$

Note that this velocity field has a net circulation: $\oint \mathbf{v}\cdot d\mathbf{l} = \int v_\phi r d\phi = K$, if the integral is taken around any closed curve that encloses the $r = 0$ axis.

Now, at the surface of the cylinder, $v_r = 0$ and

$$v_\theta = -2U\sin\theta - \frac{K}{2\pi a}$$

so that the sign of K – the intrinsic circulation associated with the $r \le a$ region – determines whether the v_θ flow outside agrees with K or with the ambient flow U. We also see from (4.17) that the flow is no longer

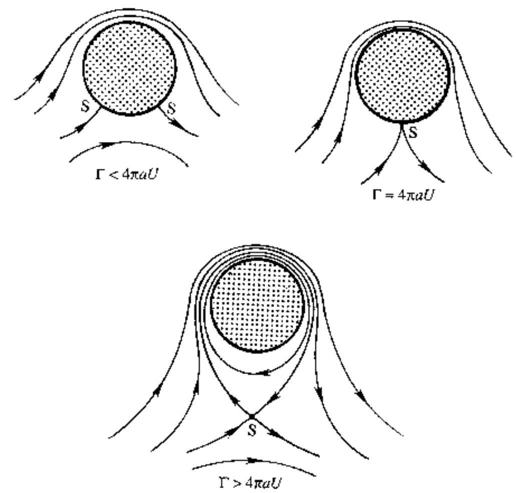

Figure 4.1. Potential flow past a circular cylinder for different values of the circulation Γ (K in our notation). Note the location of the stagnation points S depends on the ratio of circulation to flow speed U. From Kundu figure 6.14

symmetric above and below the cylinder; the flow simply moves faster "over the top". Figure 4.1 illustrates various possibilities. At low K values, there are two stagnation points, at the surface of the cylinder. At high enough K, there is one stagnation point, well below the cylinder. The streamline passing through this stagnation point contains a region of fluid which remains separated from the outer flows; it simply circulates and never reaches large distances away.

And now: there is a net surface pressure. We can find it as usual from Bernoulli, and show that it contributes a net lift:

$$L = -\int_0^{2\pi} p(a,\theta)\sin\theta a d\theta = \rho U K \tag{4.18}$$

We can remember the direction of the force by noting that the lift L in this example is upwards. If we define the direction of the circulation $\mathbf{K}$ as along the axis of the cylinder, in the sense give by the right hand rule, then the direction of the lift is $\mathbf{L} \propto \mathbf{U}\times\mathbf{K}$. This allows a reasonable analogy with basic EM: a current $\mathbf{I}$ in an external magnetic field $\mathbf{B}$ feels a force $F \propto \mathbf{I}\times\mathbf{B}$.

It turns out that this result – $L = \rho U K$ – holds for any two-dimensional shape, not just a true cylinder. A rotating body feels a lateral force: this is the *Magnus effect*. Why is this different from the potential flow solutions of chapter 3? One answer would simply be, "Bernoulli"; the asymmetric velocity above and below the rotating object leads to an asymmetric pressure, thus a net lateral force. But one could also ask how a rotating body develops the circulation which is needed for the Magnus effect. The answer here is, again, viscous effects at the boundary of the cylinder. Any rotating body must have a no-slip surface; this will generate a small layer of fluid rotating with the body.

E. Vortex lines and their behavior

Vortex lines demonstrate very interesting behavior. In these notes we only summarize; the literature is extensive here.

• **Free vortex lines** are linear (but not necessarily straight) structures, with a net circulation $\mathbf{K}$, which is constant along the line and also a constant of the motion (from Kelvin's theorem). By Kelvin's theorem, a loop in the fluid which at one point encloses the core ($r \to 0$) will have a finite circulation K; it must retain that K value no matter how the fluid is distorted in later flow. Thus, a vortex line must move with the fluid; the fluid is effectively attached to the vortex, particularly to its core (about which the line integral defining $\mathbf{K}$ is finite).

• A **vortex sheet** is made up of an infinite number of vortex filaments placed side by side, with all filaments rotating in the same sense. This sheet generates a tangential velocity discontinuity, as in Figure 4.2. Such structures are important in understanding the flow over aircraft wings.

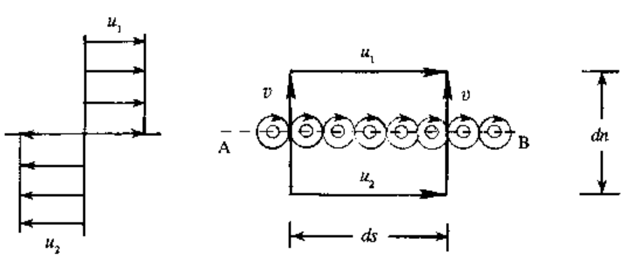

Figure 4.2. An idealized vortex sheet. From Kundu figure 5.15

• **The behavior of vortex lines** can be very intriguing. They move under their own power. A vortex line will move in a transverse flow, in a direction $\mathbf{U} \times \mathbf{K}$. (Recall the Magnus effect). A bent vortex line will move in a direction $\mathbf{K} \times \hat{\mathbf{n}}$, if $\hat{\mathbf{n}}$ is the direction of its radius of curvature R; Batchelor shows its speed $\sim K/4\pi$.

• **Images** are useful in calculating the behavior of vortex lines near boundaries (remember your images in electrostatics?). A wall, such as in Figure 4.3, is a streamline, and must have zero normal velocity. This means we can add an imaginary image vortex, behind the wall and paired with the real one, to understand the motion of the vortex line.

• **Paired vortices interact**. Two nearby vortex lines, with *opposite* $\mathbf{K}$ values, will drift together in a

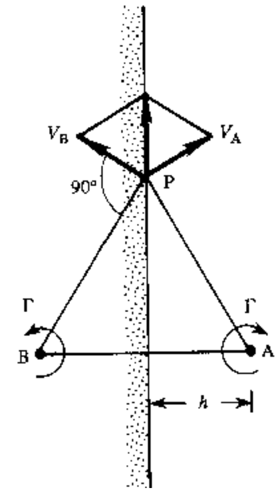

Figure 4.3. A line vortex (A) near a wall, and its image (B). From Kundu figure 5.13

direction perpendicular to their separation. If the two lines have *parallel* $\mathbf{K}$ values, they will circle around each other – or twist into a helix if one end of each is somehow tied down.

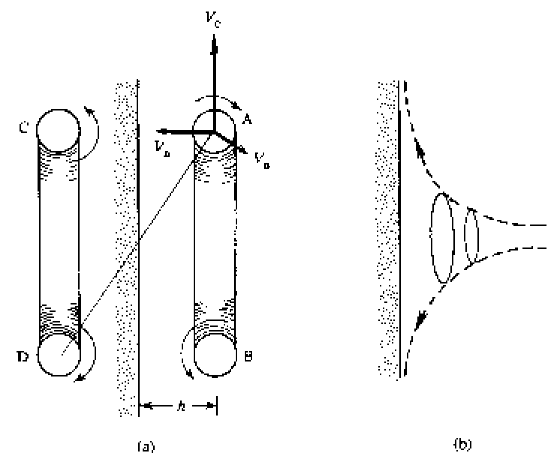

Figure 4.4. A vortex ring approching a wall. From Kundu figure 5.14

• **Vortex rings** are formed when a vortex line connects back on itself. A vortex ring will move itself along through the background fluid, at a speed $\sim K/4\pi b$ if b is the radius of the ring. (Any two opposite points around the ring act like paired, anti-parallel $\mathbf{K}$ lines. Alternatively, this is a case of well-defined curvature of a single vortex line.) One vortex ring will spread out as it approaches a wall. Two rings will move around and through each other – Faber shows this in his figures.

• **Vortex lines can reconnect**. Viscosity breaks the conservation of circulation, and allows adjacent vortex lines or rings to reconnect, thereby changing their topology.

F. Generation of Vorticity

We argued above that circulation is conserved; this was Kelvin's theorem. How, then, can a fluid which starts irrotational, ever gain vorticity? The answer, once again, is the combination of viscosity and boundary effects. To illustrate, I combine a figure from Kundu with the discussion from Batchelor.

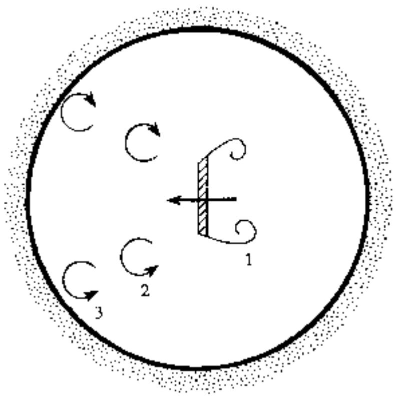

Figure 4.5. Top view of a vortex pair generated by moving the handle of a spoon through a cup of coffee. From Kundu figure 5.12

> Start with a quiescent cup of coffee, turn your spoon upside down, and move its handle perpendicular to itself through the cup. Vortices will be generated, as shown. Why? Even though there is a formal no-slip condition at the surface of the handle, there must be a finite tangential velocity somewhere close to the handle. In this problem, it's at the edges. As Batchelor says, "the motion that would be generated from rest in the absence of diffusion of vorticity across the boundary . . .is accompanied by a nonzero tangential relative velocity at the boundary. Since the no-slip condition requries the tangential component of relative velocity to be zero at each point of the solid boundary, however small the viscosity may be, the vorticity in the flow is infinite at the boundary." Thus: the need to go from a no-slip boundary to a tangential slip in the coffee, some tiny distance away, generates vorticity local to the handle; diffusion (as per 4.6) takes care of the rest.

References

Vorticity is a large, interesting and complex topic. I've mostly followed Faber and Tritton (for the words) and Kundu (for the math); but there's also good material in Shivamoggi, not to mention a whole book by Saffman (quite mathematical), and several useful articles in *Annual Review of Fluid Mechanics* over the years.