5. LAMINAR FLOW: MORE APPLICATIONS

We continue the previous discussion with more examples and applications from here and there. We start with a simple discussion of flow in a (quasi) two-dimensional, rotating system – all sorts of fun things happen when you're rotating. We then look at two more mathematically involved examples of viscous flows.

A. Geostrophic Flow

What happens in a rotating system? Because our important application is to terrestrial (geophysical) flows, we can work in the incompressible limit. Referring back to chapter 1, the general equation of motion comes from (1.20), but now has viscosity added, as per Chapter 2:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \Omega^2\mathbf{R} - 2\mathbf{\Omega}\times\mathbf{v} + \nu\nabla^2\mathbf{v} \quad (5.1)$$

We consider the limit which is interesting for terrestrial problems: steady (or nearly steady) flow, in which the advective term ($\mathbf{v}\cdot\nabla\mathbf{v}$) and the viscous term ($\nu\nabla^2\mathbf{v}$) are small compared to the Coriolis force ($2\mathbf{\Omega}\times\mathbf{v}$). In addition we ignore the centrifugal force and gravity terms.[1] We then have a situation in which the equation of motion simplifies to

$$2\rho\mathbf{\Omega}\times\mathbf{v} + \nabla p = 0 \quad (5.2)$$

This is known as *geostrophic flow*; such flows have interesting properites, as follows.

• The velocity is perpendicular to the pressure gradient. Thus, isobars are also streamlines; the flow is along lines of constant pressure. This is in marked contrast to non-rotating systems, where we tend to think of pressure gradients driving the flows, or (from Bernoulli's equation) think of pressure variations along streamlines.

• The flow must be two-dimensional. That is: take the curl of (5.2), to get $\nabla\times(\mathbf{\Omega}\times\mathbf{v}) = 0$. But if $\mathbf{\Omega} \parallel \hat{\mathbf{z}}$, say, this requires that $d\mathbf{v}/dz = 0$. The flow cannot depend on z. This is known as the Taylor-Proudman theorem, and will be important in our discussion of Rossby waves, below. It has one striking consequence: the Taylor column. Think about a rock on the bottom of a tank of water. You tie a string to the rock and pull it through the water. If the tank is static, the water well above the rock is not very disturbed by the rock's motion (that is, the flow will vary with height, above the rock). Now consider the same thing but with a rotating water tank. The Taylor-Proudman result says that the flow at all heights above the rock must be the same. This produces a Taylor column: the water above the rock forms a vertical column which is fixed relative to the rock; water not over the rock flows around this column, as though it were a solid object.

ATMOSPHERIC CIRCULATION

One striking example of our first point is the atmospheric circulation around a pressure extremum: the circulation is counterclockwise (seen from above) around a low, and clockwise around a high – as shown in the figure.

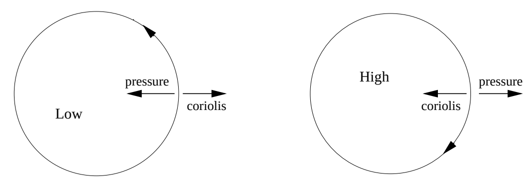

Figure 5.1. Illustrating geostrophic flow around high and low pressure centers. Following Kundu Figure 13.4.

The *jet stream* is a related effect. To be specific, think about the *(north) polar jet stream*. The atmosphere at the top of the world is cold, while that at lower latitidues is warm; the hot and cold air masses meet in a region called the *polar front*. The horizontal pressure gradient across the polar front generates transverse winds, again by the Coriolis balance of (5.2); this large-scale circulation, from west to east, is called the polar jet stream. It typically lies at about 10 km altitude, has average speed $\sim$ 100 km/s, but can reach maximum speeds 300-400 km/s. The northern hemisphere also has a sub-tropical jet, where the temperate mid-latitude air meets the hot, tropical air. These two jets are by no means steady – they wander about, and sometimes split, depending on the local thermodynamics of the atmosphere. Some of the wiggles and fluctuations in the jet streams are the result of fluid instabilities – which we'll discuss later in the course.

ROSSBY WAVES

These waves exist only in particular geometries in a rotating system. For a simple version, consider a rotating layer of fluid, bounded by two planes which are not quite parallel. As in Figure 5.2, let the height of the fluid (parallel to the rotation axis $\hat{\mathbf{z}}$) vary slightly. We know from the Taylor-Proudman theorem that the fluid flow must be independent of z. A "geostrophic" flow

[1] Purists can note that both of these can be expressed as gradients of a scalar: $\mathbf{g} = \nabla\Phi_g$, and $2\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = -\nabla(\Omega^2R^2)$ (in the notation of §1.D.). Thus they can be subsumed into the pressure term – where they will be only small corrections.

in this geometry would have to follow contours of constant depth h. Consider, then, a small radial displacement of a column of the fluid, to a region with $h + \delta h$. This violates the T-P theorem; the column will try to return to its initial position. But it has some inertia, so it will overshoot. We thus have an oscillation: this is one version of a Rossby wave.

Rossby waves exist in the atmosphere, and in the ocean; they are found to be low frequency $\omega < \Omega$ and long wavelength. The restoring force for terrestrial waves is due to variations in the depth of the atmosphere parallel to $\boldsymbol{\Omega}$ (here meaning the earth's rotation). In the atmosphere they are easily observed as the large-scale meanders of the mid-latitude jet stream. In the ocean they are harder to find, being very small amplitude (several cm) and very long wavelength (hundreds of km), but now detected in satellite data.

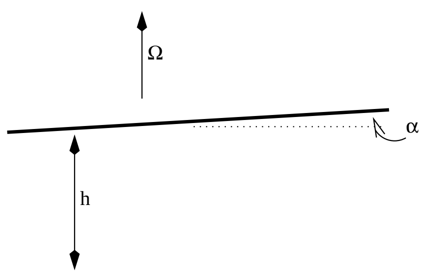

Figure 5.2. The geometry for a simple version of Rossby waves. The curved arrow marks the deviation of the upper plate from parallel. Following Tritton Figure 15.13.

THE EKMAN LAYER

Coriolis forces also do interesting things at boundaries. The two applications relevant to geophysical flows are solid boundaries (such as land) and free boundaries (such as the ocean). In each case, the combination of a driving wind (think of geostrophic flow)and viscosity within the boundary layer results in a transverse flow within that layer. In this section I present an overview; details will appear in the homework.

• **Solid boundary**. At some high altitude, let the wind velocity be $U\hat{\mathbf{x}}$. Thus, (5.2) reduces at high altitudes to

$$2\Omega U = -\frac{1}{\rho}\frac{dp}{dy}$$

(note $dp/dx = 0$, right?). Within the boundary layer we must keep the viscous terms. Here, (5.1) becomes

$$\begin{aligned} -2\Omega v_y &= \nu \frac{d^2 v_x}{dz^2} \\ 2\Omega v_x &= \nu \frac{d^2 v_y}{dz^2} - \frac{1}{\rho}\frac{dp}{dy} \end{aligned} \qquad (5.3)$$

The boundary conditions are: $\mathbf{v} = 0$ at $z = 0$ (no-slip at the solid surface), and $\mathbf{v} = U\hat{\mathbf{z}}$ as $z \to \infty$. This system solves to

$$\begin{aligned} v_x &= U\left[1 - e^{-z/\delta}\cos(z/\delta)\right] \\ v_y &= Ue^{-z/\delta}\sin(z/\delta) \end{aligned} \qquad (5.4)$$

with $\delta^2 = \nu/\Omega$. Thus, the velocity vector rotates through a spiral, while it increases from zero at the ground to $\sim U\hat{\mathbf{x}}$ at $z \gtrsim \delta$.

• **Free boundary**. Our driver here is the wind stress at the ocean surface, again in the x direction; call it $\tau\hat{\mathbf{x}}$. We can ignore pressure gradients in this case; thus our equations are, simply,

$$\begin{aligned} -2\Omega v_y &= \nu \frac{d^2 v_x}{dz^2} \\ 2\Omega v_x &= \nu \frac{d^2 v_y}{dz^2} \end{aligned} \qquad (5.5)$$

Take $z = 0$ at the ocean surface, and negative below. Our boundary conditions now are $\mathbf{v} \to 0$, $z \to -\infty$, and $dv_y/dz = 0, dv_x/dz = \tau/\rho\nu$, $z = 0$ (refer back to the definitions of stress in chapter 4). These solve to

$$\begin{aligned} v_x &= V_o e^{-z/\delta}\cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right) \\ v_y &= -V_o e^{-z/\delta}\sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right) \end{aligned} \qquad (5.6)$$

where $V_o = \tau/\rho\sqrt{2\Omega\nu}$, and δ is the same as above. Again, the velocity vector rotates through a spiral; note that it is at a $45°$ angle to $\vec{\tau}$ at the ocean surface.

Both cases have similar consequences. A prevailing wind or flow results in a transverse flow within the Ekman boundary layer (this is *Ekman transport)*. Mass conservation then forces a net vertical motion (why?) into or out of the layer (this is *Ekman pumping*). In the atmosphere this is related to updrafts in low pressure regions (thus storm formation); in the ocean this is related to phenomena such as upwelling of cold subsurface water in regions with a steady prevailing wind direction.

B. Viscous Flow: Time-Dependent Problems

Viscosity is dissipative: we expect a viscous flow to decay with time. In this section we consider solutions of the basic equations, (4.5) or (4.6), but here omitting gravity and the advective terms. Thus we want solutions of

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \nabla p = \rho\nu\nabla^2\mathbf{v} \qquad (5.7)$$

and of

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \nabla^2 \boldsymbol{\omega} \tag{5.8}$$

The ∇^2 term makes these "diffusive" DE's, and we can use standard methods for their solution. Let's do this by example.

1. SIMILARITY METHODS IN A DIFFUSION EQUATION

Here's one good approach to solving diffusion equations. Take a Cartesian system, ignore the pressure gradient, and take $\mathbf{v} = v(y,t)\hat{\mathbf{x}}$ as a simple geometry. The basic equation is then

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2} \tag{5.9}$$

To set up a *similarity transform*, we note that there is no physical scale applied to this problem (no edges of finite length, no rocks in the flow). We also notice that the quantity $\eta = y/\sqrt{\nu t}$ is dimensionless. We therefore guess that the (normalized) solution, $v(y,t)/U$ can be expressed as a function of η alone, call it $f(\eta)$. Now do some chain rules (you should check this!)

$$\frac{\partial v}{\partial t} = -\frac{U}{2}\frac{\eta}{t} f' \; ; \quad \frac{\partial v}{\partial y} = \frac{U}{2}\eta f' \; ; \quad \frac{\partial^2 v}{\partial y^2} = \frac{U}{4}\eta^2 f'' \tag{5.10}$$

(here, $f' = df/d\eta$, and etc). Putting these into (5.9), our DE becomes simpler, and can be integrated by simpler methods:

$$f'' + 2\eta f' = 0 \; ; \quad f(\eta) = A \int e^{-\eta'^2} d\eta' + B \tag{5.11}$$

and the integration constants A, B (and/or limits of the integral) are chosen for the boundary conditions of the problem.

2. SMOOTHING OUT A VELOCITY JUMP

Consider planar flow with a sharp jump in the velocity. At $t = 0$, take

$$v_x = +U \, , \qquad y > 0$$
$$= -U \, , \qquad y < 0$$

Viscosity will clearly try to smooth out this jump. We again have (5.9) as the basic equation; and (from the last section), we can find its solution, with reasonable boundary conditions $v_x \to \pm U, |y| \to \infty$):

$$\begin{aligned} v(y,t) &= \frac{U}{(\pi \nu t)^{1/2}} \int_0^y e^{-u^2/4\nu t} du \\ &= U \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) \end{aligned} \tag{5.12}$$

The last expression simply notes that this defines the error function, $\mathrm{erf}(x)$. Thus, the $|y| \to 0$ region obeys $v_x \propto y$, and the outer has $v_x \to \pm U$, as required; the transition occurs at $y \sim \sqrt{4\nu t}$. THus, viscosity spreads the transition region out with time.

3. FLOW ABOVE AN OSCILLATING PLATE

Another variant is an infinite, flat plate which moves back and forth parallel to itself. The basic equation for fluid above the plate is again (5.9), and the boundary conditions are $v(0,t) = U\cos\omega t$, and that v stays bounded as $y \to \infty$. To solve this, we try a separable solution, $v(y,t) = e^{i\omega t} f(y)$. Note, we are looking for the "steady" solution, established after initial transients have gone away. Plugging this test solution in and doing algebra, we find

$$v(y,t) = U e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) \tag{5.13}$$

The cosine term here represents a signal propagating away in the y direction, while the exponential is a decay in the y direction. Thus, the flow resembles a damped transverse "wave", with wavelength $2\pi\sqrt{2\nu/\omega}$ in the y direction, propagating in the y direction, with amplitude falling off as $e^{-y\sqrt{\omega/2\nu}}$.

4. IRROTATIONAL VORTEX DECAY

A third application is the time-dependent evolution of the irrotational vortex. Switching to cylindrical coordinates, we have

$$\frac{\partial v}{\partial t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv)\right] \tag{5.14}$$

The interesting solution of this equation, for a system which initially obeys $v = K/2\pi r$ everywhere, is

$$v(r,t) = \frac{K}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right) \tag{5.15}$$

Thus: we find $v \propto r$ for $r \to 0$ (that is the core of the vortex); $v \propto 1/r$ for $r \to \infty$ (the outer part of the vortex); and the transition occurs at a radius which grows with time, as $r \propto \sqrt{4\nu t}$. Thus, viscous dissipation causes the core of the vortex to spread. Some details of this may appear in the homework.

C. Creeping Flow

Now, consider very viscous flow; flow at very low Reynolds number. This is called *creeping flow*.[2] In creeping flow we assume the inertial term $\mathbf{v}\cdot\nabla\mathbf{v}$ can be dropped compared to the viscous and pressure gradient terms: this means the flow speed is *very* slow. We continue to assume steady state and incompressible flow. The basic equaiton is, then,

$$\nabla p = -\rho\nu\nabla\times\boldsymbol{\omega} = \rho\nu\nabla^2\mathbf{v} \tag{5.16}$$

Taking the curl of this, recalling $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the vorticity, we have

$$\nabla^2\boldsymbol{\omega} = 0 \tag{5.17}$$

Now, we consider flow around a sphere of radius a, placed in a stream flow with asymptotic velocity U. We want to find the drag force between the sphere and the rock. The solution is long but straightforward ... here goes.

First, we need to find the velocity field. Pick polar coordinates so that $\theta = 0$ corresponds to the direction of downstream flow. The only nonzero component of ω is then

$$\omega_\phi = \frac{1}{r}\left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right] \tag{5.18}$$

But this is axisymmetric flow, so we can use a stream function (as in Chapter 2); in terms of ψ, the vorticity is

$$\omega_\phi = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \tag{5.19}$$

Putting this into (5.32), we get a fourth order equation for ψ:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = 0 \tag{5.20}$$

This is now something can be solved. Our boundary conditions are

$$\begin{aligned} \psi(a,\theta) &= 0 \\ d\psi/dr(a,\theta) &= 0 \\ \psi(\infty,\theta) &= \frac{1}{2}Ur^2\sin^2\theta \end{aligned} \tag{5.21}$$

[2] For once the name makes sense!

The first is $v_r = 0$ at the surface (recall $v_r \propto d\psi/d\theta$, and we want $v_r = 0$ for all θ); the second is $v_\theta = 0$ at the surface; and the third is uniform flow at infinity. Choosing a separable solution, $\psi(r,\theta) = f(r)\sin^2\theta$ (to match the distant boundary), we find the solution is

$$\psi(r,\theta) = Ur^2\sin^2\theta\left(\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right) \tag{5.22}$$

From the stream function, we can find the velocity components:

$$\begin{aligned} v_r &= U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) \\ v_\theta &= -U\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{2r^3}\right) \end{aligned} \tag{5.23}$$

We'll need these in the next step.

Next, we want to find the drag force on the sphere – which is the integral of the stress over the surface of the sphere. Separating out Σ_{ij} as the non-pressure component of the stress tensor, the component of the drag force in the $\mathbf{U}$ direction[3] is

$$F_D = [-p\cos\theta + \Sigma_{rr}\cos\theta - \Sigma_{r\theta}\sin\theta]_{r=a} \tag{5.24}$$

For the pressure, we go back to (5.16) and solve for p (taking zero pressure at infinity, or solving for the overpressure):

$$p(r,\theta) = -\frac{3a\rho\nu U\cos\theta}{2r^2} \tag{5.25}$$

(Note that we couldn't use Bernoulli's law here; why?) Thus, the pressure is highest at the forward stagnation point, and lowest (negative in fact) at the rear stagnation point. For the rest of the stress tensor, we use the velocity solution, from (5.23), to get

$$\begin{aligned} \Sigma_{rr} &= 2\nu\frac{\partial v_r}{\partial r} = 2\nu U\cos\theta\left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right) \\ \Sigma_{r\theta} &= \nu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right] = -\frac{3\nu Ua^3}{2r^4}\sin\theta \end{aligned} \tag{5.26}$$

Putting this into the expression for f_D, multiplying by $4\pi a^2$ and integrating over θ, gives the drag

$$F_D = 6\pi\rho\nu aU \tag{5.27}$$

[3] Compare our earlier discussion in Chapter 3, in and around (3.24).

Thus, we get a finite drag due to the viscosity. This is *Stokes' law*.

References

I've mostly followed Kundu, who does the more mathematical examples. He also has a good chapter on geophysical fluid dynamics. Tritton is also good for geostrophic flows – as always his discussions are helpful (not just pure math).