

6. BASICS OF COMPRESSIBLE FLOW

Up to now we have assumed incompressible flows. We now extend to the more general case, when the density is allowed to vary. This is necessary to work with (i) wave propagation; (ii) hydrostatic atmospheres, over scales large compared with the scale height; (iii) flows in which the fluid speed is comparable to (or larger than) the sound speed.

A. Some useful thermodynamic quantities

Since density changes are accompanied by temperature changes, we need to remember a bit of thermodynamics. I store some terms and definitions here, in no particular order (some of the definitions are circular...). When specific expressions are needed, I work with an ideal gas.

- Equation of state. $p = nk_B T = \rho R T$; k_B is the Boltzmann constant, a fundamental constant. R is the gas constant, which varies by composition. It's related to the "fundamental" gas constant \mathcal{R} by $R = \mathcal{R}/M$, if M is the molecular/atomic weight of the gas in question. Or if you prefer microphysics, R is related to the Boltzmann constant by $R = k_B/m$, if m is the mass per particle.

- Internal energy and enthalpy. Let e be the internal energy (per mass): we define the enthalpy as $h = e + p/\rho$. For a simple ideal gas we have $p = nkT$; $e = kT/m$ (if m is the mass per particle); also $\rho e = p/(\gamma - 1)$; and $\rho h = \gamma p/(\gamma - 1)$.

- Specific heats and adiabatic index. The SH's are defined by $de/dT = c_V$ (constant volume) and $dh/dT = c_P$ (constant pressure). The adiabatic index can be defined as $\gamma = c_P/c_V$, the ratio of specific heats. We also have $c_p - c_V = R$; $c_v = R/(\gamma - 1)$; $c_p = \gamma R/(\gamma - 1)$.

- Degrees of freedom and adiabatic index. Let f be the number of d's of f, so that the mean KE per particle is $(f/2)k_B T$. This connects to the adiabatic index by $\gamma = (f + 2)/f$. Typical values for γ are $\gamma = 1.4$ for the atmosphere; $\gamma = 5/3$ for a monatomic ideal gas; and $\gamma = 4/3$ for a relativistic gas (such as a photon gas).

B. Hydrostatics: gaseous atmospheres

The effects of compressibility must be considered in most situations involving gases (or plasmas) in static equilibrium in a gravitational field.

1. CONSTANT GRAVITY: THE EXPONENTIAL ATMOSPHERE

One example is hydrostatic balance in a fixed (external) gravitational field \mathbf{g} . The most obvious application of this is the earth's atmosphere. The basic equation is:

$$\nabla p = \rho \mathbf{g} \quad (6.1)$$

Or in a vertically stratified medium, such as earth's atmosphere, this is

$$\frac{dp}{dz} = \rho g \quad (6.2)$$

and g is constant over any relevant scale in this application. We can directly define a *scale height* from this. If T is also constant with height, the pressure solution is

$$p(z) = p_o e^{-z/H} \quad (6.3)$$

where $H = RT/g = kT/mg$ is the scale height.

Another example of this, perhaps less obvious, is the interstellar medium in our galaxy. For non-astronomers: our galaxy is a flat disk, composed of stars and gas rotating about a common center. The thickness of the disk is controlled by the local mass density (which produces a constant gravity, \mathbf{g}), and the temperature (plus random motions) of the gas or stars in the disk. The vertical structure of the disk is controlled by the same equation, (6.3) ... just on a much larger scale.

2. VARIABLE GRAVITY: THE ISOTHERMAL SPHERE

A different HSEq system is a *self-gravitating sphere*. This might be a naive picture of a star: held together by its own gravity, and supported against gravitational collapse by its internal energy. The basic HSEq equation, (6.1) still holds; in spherical geometry it is

$$\frac{dp}{dr} = -\rho \frac{GM(r)}{r^2} \quad (6.4)$$

where $M(r)$ is the mass inside r . Alternatively, this can be written

$$\frac{dp}{dr} = \rho \nabla \Phi_g \quad (6.5)$$

where Φ_g is the gravitational potential, defined through $\mathbf{g} = -\nabla \Phi_g$. The mass, density and Φ_g are connected via

$$\begin{aligned} M(r) &= \int_0^r 4\pi r'^2 \rho dr' ; \\ \nabla^2 \Phi_g &= \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho \end{aligned} \quad (6.6)$$

Putting this into (6.5) and differentiating with r , the basic equation becomes

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{d\rho}{dr} \right) = -\frac{G}{RT} 4\pi r^2 \rho \quad (6.7)$$

The solutions of this are less straightforward. First, we note that a basic scale length appears:

$$a_o = \left(\frac{9RT}{4\pi G \rho_o} \right)^{1/2} \quad (6.8)$$

if ρ_o is some characteristic density, say that at the origin. (the numerical constants appear to simplify things later on.) We expect physical solutions of (6.7) to involve lengths scaled to a_o .

Consider an isothermal gas; these are the simplest solutions. One solution of (6.7) is the simple power law, $\rho \propto 1/r^2$. The divergence as $r \rightarrow 0$ keeps this from being an interesting solution, however. More interesting physical solutions must be found numerically, starting with $\rho(r=0) = \rho_o$ and working out. These solutions do, indeed, have a turnover at $r \simeq a_o$; at large radii they do approach $\rho \propto 1/r^2$. (These solutions also have problems, for the total mass $M(r)$ diverges $\propto \ln r$ as $r \rightarrow \infty$. Physically, the problem is that the gas cannot be maintained isothermal everywhere.)

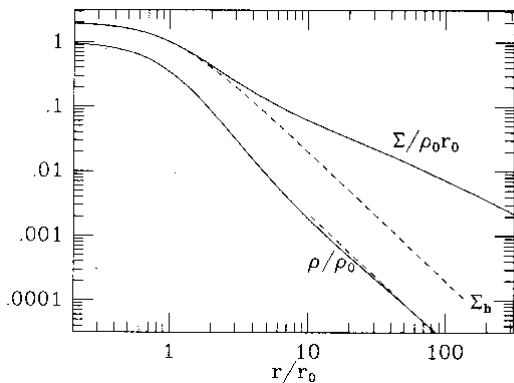


Figure 6.1. Density (and projected density, Σ) solution for a self-gravitating isothermal sphere. The dotted line is the asymptotic $\rho \propto 1/r^2$ power-law solution. From Binney & Tremaine figure 6.7

Self-gravitating spheres can also be modelled as adiabatic gases. Take the equation of state as $p = K\rho^\gamma$. The hydrostatic equation, (6.5), and the differential form (6.7), become

$$\begin{aligned} K\gamma\rho^{\gamma-2} \frac{d\rho}{dr} &= -\rho \frac{d\Phi}{dr}; \\ \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) &= 4\pi G\rho \end{aligned} \quad (6.9)$$

Numerical solutions of this equation are called *polytropes*, or *Lane-Emden solutions*, for polytropic index $n = 1/(\gamma - 1)$. These are physically well-behaved, in that the density is finite at the center and falls faster than $1/r^2$ at large r .

3. REALITY: NONISOTHERMAL ATMOSPHERES

Just a note of caution here. Both of the preceding examples assumed a constant temperature throughout the atmosphere. This naive assumption allowed us to find (nearly) analytic solutions. More realistically, however, we expect the temperature to be a function of position, controlled by the thermodynamics of the system. For earth's atmosphere, $T(z)$ is determined by the combined actions of solar radiation coming in at the top, all the chemistry and energy transfer effects within the atmosphere, and radiative losses back into space. For a star, $T(r)$ is determined by nuclear energy generation in the star's core, radiative energy transfer within the star, and eventual radiative losses into space.

4. ADIABATIC ATMOSPHERE

Another analytic model of an atmosphere assumes the gas is adiabatic: that is, as it compresses or expands (while satisfying HSEq), it heats or cools accordingly. This is useful in analyzing the stability of an atmosphere to convection (which we'll do immediately below).

To develop this idea, go back to basic HSEq, (6.1): $dp/dz = -\rho g$. But now, assume the gas is adiabatic, so that

$$\frac{T}{T_o} = \left(\frac{p}{p_o} \right)^{(\gamma-1)/\gamma}; \quad \frac{\rho}{\rho_o} = \left(\frac{p}{p_o} \right)^{1/\gamma} \quad (6.10)$$

We can use these to relate dT/dz to dp/dz for an *adiabatic atmosphere*:

$$\frac{1}{T} \frac{dT}{dz} = \frac{\gamma - 1}{\gamma} \frac{1}{p} \frac{dp}{dz}; \quad \frac{1}{\rho} \frac{d\rho}{dz} = \frac{1}{\gamma} \frac{1}{p} \frac{dp}{dz} \quad (6.11)$$

From here, assuming $p = \rho RT$ (ideal gas) and using $c_p - c_v = R$ (refer back to §6.1) we get the condition for an adiabatic atmosphere:

$$\frac{dT_{ad}}{dz} = -\frac{g}{c_p} = -\frac{(\gamma - 1) mg}{\gamma k_B} \quad (6.12)$$

where c_p is the isobaric specific heat, and m is the mass per particle. This clearly gives a linear temperature drop with altitude z ; $g/c_p \simeq 10^\circ\text{C}/\text{km}$ for typical atmospheric conditions. The vertical temperature gradient is often called $\Gamma = dT/dz$; and is sometimes called the "lapse rate".

C. Convective Stability

So ... we now have a description of hydrostatic atmospheres. But, are they stable? One of the most important instabilities, for the earth's atmosphere or for a star, is convection. We need to understand whether the atmosphere is stable, or unstable, to convection – that is, whether or not convection will develop spontaneously.

To picture the situation, here's how Shore puts it:¹

Picture a duck sitting calmly on a pond. . . If we say that the bird is buoyant, we mean that if we depress him a bit by pushing from above, he will bob back to the surface and, ignoring his agitation, bounce up and down for awhile [this is called *neutral bouyancy*]. . . If we have one, on the other hand, who is not well preened and therefore not waterproof, and [we] push down on him, he may sink. Now think of a blob which is hotter than its surroundings. It will begin to rise, since we already know that its density will be lower than that of the [surrounding] medium, and it will thus be bouyant. If it remains underdense, it will continue to rise – we call this an instability. It will continue to rise until it reaches a level at which it is neutrally bouyant again. On the other hand, if the blob is pushed down, and if it remains overdense, it will sink until it reaches a point at which the density again allows for stable balance.

We would like to find a condition to tell if the situation is stable or unstable. To get there, think about some blob again: assume it starts at some vertical position z , with density and pressure in balance with its surroundings (*i.e.*, the “outside”, which usually refers to the atmosphere). Thus: it starts at $\rho_{in} = \rho_{out} = \rho_1$ and $T_{in} = T_{out} = T_1$. Now, raise it some distance Δz ; let the conditions in the blob at $z + \Delta z$ be labelled by “*”. For the blob, we assume it evolves *adiabatically*. Thus, it reaches a new density and temperature,

$$\rho_{in}^* = \rho_1 + \left(\frac{d\rho}{dz} \right)_{ad} ; \quad T_{in}^* = T_1 + \left(\frac{dT}{dz} \right)_{ad} \quad (6.13)$$

If we want to evaluate the z -derivatives in (6.13), we need to know *something* about the specific situation. A common assumption is that the blob rises slowly enough to remain in pressure balance with its surround-

ings. If this is the case, we will have

$$p_{in}^* = p_{out}^* ; \quad \rho_{in}^* = \rho_{in} \left(\frac{p_{in}^*}{p_{in}} \right)^{1/\gamma} ; \quad T_{in}^* = T_{in} \left(\frac{p_{in}^*}{p_{in}} \right)^{1/(\gamma-1)} \quad (6.14)$$

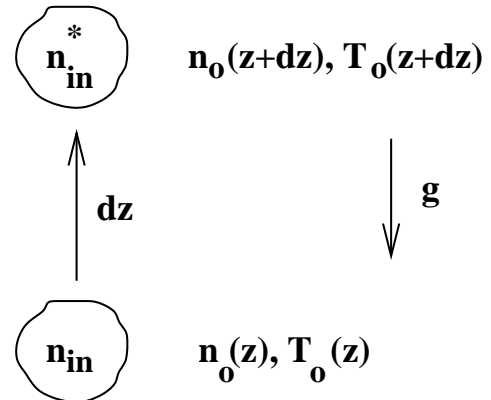


Figure 6.2. A cartoon illustrating the buoyant/convective instability. Imagine a blob starts in balance with its surroundings. It is then displaced vertically, some dz ; during this rise it (i) remains in pressure balance with its surroundings, and (ii) remains adiabatic. How does its density at $z + dz$ compare to the density outside? The answer to this question determines the stability of the atmosphere.

The surroundings, however, are not necessarily adiabatic: they have some other dT/dz and $d\rho/dz$ values (specified by the situation – for instance the heating/cooling balance for the outer layers of a star, or the earth's atmosphere). So, we can determine stability or instability by asking whether, when the blob has risen this Δz , it is at a *higher* or *lower* density than the surrounding atmosphere. In the first case it will sink again (and thus the atmosphere is stable); in the second case it will keep rising (and thus the atmosphere is unstable). Thus, our condition for instability becomes condition on the external density gradient. If we assume the blob remains in pressure balance with its surroundings, we also have a condition on the atmospheric temperature gradient. Thus, the atmosphere is buoyantly *unstable* if

$$\left(\frac{d\rho}{dz} \right)_{ad} < \left(\frac{d\rho}{dz} \right)_{atm} ; \quad \left(\frac{dT}{dz} \right)_{ad} > \left(\frac{dT}{dz} \right)_{atm} \quad (6.15)$$

Because we usually consider situations with $d\rho/dz < 0$, for instance a hydrostatic atmosphere, the condition for *instability* is often written in terms of absolute values:

$$\left| \frac{d\rho}{dz} \right|_{ad} > \left| \frac{d\rho}{dz} \right|_{atm} ; \quad \left| \frac{dT}{dz} \right|_{ad} < \left| \frac{dT}{dz} \right|_{atm} \quad (6.16)$$

¹ *An Introduction to Astrophysical Hydrodynamics* (Academic Press) 1992, ch. 9.

Thus: if the outside (atmospheric) temperature changes too rapidly with altitude, the atmosphere is convectively unstable. An underdense blob will continue to rise, and an overdense blob will sink.

1. ADIABATIC ATMOSPHERE

Referring to our discussion of adiabatic atmospheres, above, we recall that the real atmosphere needs to be compared to the ideal, adiabatic case in order to determine convective (in)stability. This gives a condition for instability, written in terms of either the pressure or temperature gradients:

$$\frac{1}{\gamma} \frac{1}{p} \frac{dp}{dz} > \frac{1}{\rho} \frac{d\rho}{dz}; \quad \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \frac{dp}{dz} > \frac{dT}{dz} \quad (6.17)$$

Note that both p and T drop with z in any hydrostatic situation; so both sides of these inequalities involve negative quantities.

2. POTENTIAL TEMPERATURE

The adiabatic gradient can be important in the convective stability of the atmosphere. Another useful quantity here is the *potential temperature*. Consider a parcel of atmosphere, that starts in local balance at some (p, T) . Take it *adiabatically* to some other point, with local pressure p_s (typically sea level pressure). The temperature this parcel reaches at point s is called its potential temperature – called θ – which is given by

$$T = \theta \left(\frac{p}{p_s}\right)^{(\gamma-1)/\gamma}; \quad \frac{T}{\theta} \frac{d\theta}{dz} = \frac{d}{dz} (T_{ad} - T) \quad (6.18)$$

But this is useful, because the gradient of θ depends on the difference between the actual and adiabatic gradients in the atmosphere; thus θ is a handy tool when one's thinking about convection. Specifically, from the results above, we see that the atmosphere is convectively stable if $d\theta/dz > 0$, and unstable if $d\theta/dz < 0$.

3. BRUNT-VÄISÄLÄ FREQUENCY

Now, let's return to the blob (or the duck). If the atmosphere is convectively stable, the blob (or duck) will simply bob up and down. Its easy to show this is simply harmonic motion, and to find its characteristic frequency.

To proceed, picture the earth's atmosphere. Move the blob vertically by some Δz , so that it reaches a new density ρ_{in}^* . The blob's new density, still assuming an adiabatic displacement, will be

$$\rho_{in}^* = \rho_1 + \Delta\rho^* = \rho(z) + \frac{1}{\gamma} \frac{\rho}{p} \frac{dp}{dz} \Delta z \quad (6.19)$$

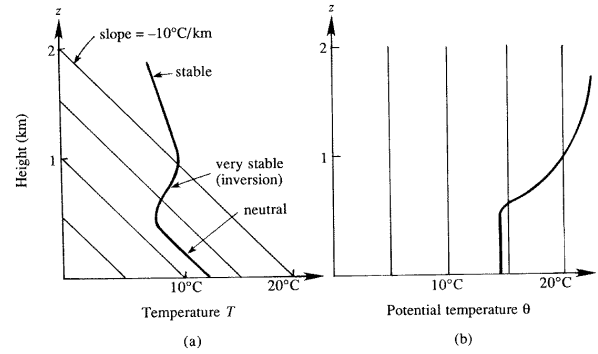


Figure 6.3. Vertical variation of the (a) actual and (b) potential temperature in a typical terrestrial atmosphere. The straight lines are the adiabatic $T_{ad}(z)$ structure. From Kundu figure 1.9.

while the ambient density has changed as

$$\rho(z + dz) = \rho(z) + \frac{d\rho}{dz} dz \quad (6.20)$$

(Note, I've now dropped the subscript "out"). Thus, the buoyant force (per unit volume of the blob/duck) is

$$F_{buoy} = g \left[\left(\frac{d\rho}{dz}\right)_{ad} - \left(\frac{d\rho}{dz}\right) \right] \Delta z \quad (6.21)$$

Alternatively, using the definition of potential temperature, this becomes

$$F_{buoy} = \frac{g}{\theta} \frac{d\theta}{dz} \Delta z \quad (6.22)$$

So: the buoyant force $\propto \Delta z$ – this is clearly simple harmonic motion. The equation of motion (still per mass) can be written

$$\frac{d^2 \Delta z}{dt^2} = \frac{1}{\rho} F_{buoy} = N^2 \Delta z \quad (6.23)$$

and this last implicitly defines the *Brunt-Väisälä frequency*, N , frequency,²

$$N^2 = \frac{g}{\rho} \left[\left(\frac{d\rho}{dz}\right)_{ad} - \left(\frac{d\rho}{dz}\right) \right] \quad (6.24)$$

If the right hand side is positive (which is the same thing as saying the atmosphere is convectively stable – refer back to 6.15), the blob simply oscillates at the B-V frequency. If the RHS is negative (if the atmosphere is unstable), the displacement Δz grows exponentially.

² Caveat to the student: very few authors leave N^2 in this form – they express it in terms of θ , dT/dz , specific heats, and what have you.

D. Energetics of Compressible Flow

We also need an equation for energy conservation. We consider two forms of energy: kinetic energy density, $v^2/2$, and internal energy density, e , both defined per unit mass.

The net energy in our volume V is $\int_V \rho(e + \frac{1}{2}v^2) dV$. The net rate of change of this energy, from intrinsic changes and from flows is

$$\int_V \frac{\partial}{\partial t} \left[\rho e + \frac{1}{2} \rho v^2 \right] dV + \int_V \nabla \cdot \left[\mathbf{v} \left(\rho e + \frac{1}{2} \rho v^2 \right) \right] dV \quad (6.25)$$

This net energy change must be accounted for by (a) work done by the external force, \mathbf{f} ; (b) work done by the external pressure; (c) direct energy gains or losses,³ which we collect as \mathcal{H} .⁴ These three energy-change factors are

$$\int_V \rho \mathbf{f} \cdot \mathbf{v} dV - \int_A p \hat{\mathbf{n}} \cdot \mathbf{v} dA + \int_V \mathcal{H} dV \quad (6.26)$$

We can use Gauss' law to express the pressure work term as a volume integral, and can derive one version of the differential energy conservation law:

$$\frac{\partial}{\partial t} \left[\rho e + \frac{1}{2} \rho v^2 \right] + \nabla \cdot \left[\mathbf{v} \left(\rho e + p + \frac{1}{2} \rho v^2 \right) \right] = \rho \mathbf{f} \cdot \mathbf{v} + \mathcal{H} \quad (6.27)$$

Note that the enthalpy $h = e + p/\rho$ appears naturally in the second term on the LHS.

At this point, we need to look at alternative forms of (6.27). The forms we derived for mass conservation (1.4) and momentum conservation (1.10 or 2.2) are pretty standard. However, there does not seem to be one standard form for the energy conservation equation; rather, one uses the form that works best in a given application. Therefore, at the expense of a little algebra, we will look at several alternate forms of (6.27).

First, we simply separate out the $\partial\rho/\partial t$ and $\nabla \cdot (\rho\mathbf{v})$ terms in (6.27), using the continuity equation (1.4), and

³ Examples of direct heating include resistive dissipation (of a current), or such things as cosmic ray heating (relevant in astrophysics); direct losses are most commonly by radiation

⁴ We will treat viscous dissipation below. We could also include thermal conductivity, which provides a separate means of energy transport out of or into the volume; it brings in second derivatives, and we will not need it here.

find

$$\rho \frac{\partial}{\partial t} \left(e + \frac{1}{2} v^2 \right) + \rho \mathbf{v} \cdot \nabla \left(e + \frac{1}{2} v^2 \right) = \rho \mathbf{g} \cdot \mathbf{v} - \nabla \cdot (p\mathbf{v}) + \mathcal{H} \quad (6.28)$$

This one alternate form that we will use again. We can isolate the rate of change of e , by subtracting $\mathbf{v} \cdot (\text{the momentum conservation equation})$ from (6.28), giving

$$\rho \frac{\partial e}{\partial t} + \rho \mathbf{v} \cdot \nabla e = -p \nabla \cdot \mathbf{v} + \mathcal{H} \quad (6.29)$$

In this expression, we can see that the rate of change of the internal energy depends explicitly on compression work (“ $p dV$ ” work), and on the net heating and cooling rates.

Yet another common form of the energy equation uses the Lagrangian derivative. Writing $\nabla \cdot \mathbf{v}$ in terms of the density derivatives, and collecting the p and ρ derivatives separately, we get

$$\frac{D}{Dt} \left(\frac{p}{\rho^\gamma} \right) = (\gamma - 1) \mathcal{H} \quad (6.30)$$

which is the last of our alternate forms of the energy equation. This last form allows us to consider a couple of important limits.

- The first is the *adiabatic limit*. If $\mathcal{H} = 0$, so that there is no net gain or loss of energy to the system, (6.30) shows that

$$\frac{p}{\rho^\gamma} = \text{constant} \quad (6.31)$$

which is the usual adiabatic law (the consequence of there being no gain or loss of heat from a system).⁵

- The second limit is the *isothermal limit*. A good many astrophysical calculations assume $T = \text{constant}$, which simplifies things enormously. From (6.22), we note that

$$p \nabla \cdot \mathbf{v} = \mathcal{H} \quad (6.32)$$

is the condition that must be satisfied if T (or e) is constant.

It might be comforting to prove that we can extract Bernoulli's relationship from this formalism, in addition to our earlier derivation from the force (Euler's) equation. We assume, again, $\mathcal{H} = 0$. We use the expression for De/Dt , and expand out the $\nabla \cdot (p\mathbf{v})$ term, using the continuity equation. We also assume the system is in a steady state, so that $\partial/\partial t = 0$; thus, D/Dt

measures the rate of change of a quantity, due to its motion through a region in which the flow field changes. We get,

$$\rho \frac{D}{Dt} \left(e + \frac{1}{2} v^2 \right) + \rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) = \rho \mathbf{g} \quad (6.33)$$

Now, using $\mathbf{g} = \nabla \Phi$, Bernoulli's relationship becomes

$$e + \frac{1}{2} v^2 + \frac{p}{\rho} + \Phi = h + \frac{1}{2} v^2 + \Phi = \text{constant} \quad (6.34)$$

which does, indeed, recover the Bernoulli relation that we derived above from momentum conservation. We recall, again, that this law holds along any one streamline in the flow.

References

I mostly follow Thompson for the basic energy conservation and dissipation analysis. The isothermal sphere discussion follows Binney & Tremaine (*Galactic Dynamics*); the terrestrial atmosphere discussion leans on Kundu and Tritton.

E. Appendix: Viscous Dissipation

Well, that was so much fun, let's do it again: *in Cartesian* to be explicit. Start with our force equation, in-

Cartesian:

$$\begin{aligned} \mathcal{D} = \frac{2\rho\nu}{3} [(D_{11} - D_{22})^2 + (D_{22} - D_{33})^2 + (D_{33} - D_{11})^2] \\ + 4\rho\nu (D_{12}^2 + D_{13}^2 + D_{23}^2) + \rho\nu_b (D_{11} + D_{22} + D_{33})^2 \end{aligned} \quad (6.39)$$

where, as above,

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \quad (6.40)$$

Cylindrical:

$$\mathcal{D} = 2\rho\nu (D_{rr}^2 + D_{\theta\theta}^2 + D_{zz}^2 + 2D_{r\theta}^2 + 2D_{\theta z}^2 + 2D_{zr}^2) + \rho(\nu_b - 2\nu/3)(\nabla \cdot \mathbf{v})^2 \quad (6.41)$$

where

$$\begin{aligned} D_{rr} = \frac{\partial v_r}{\partial r}; \quad D_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}; \quad D_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \frac{v_\theta}{r} \right) \\ D_{zz} = \frac{\partial v_z}{\partial z}; \quad D_{\theta z} = \frac{1}{2} \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right); \quad D_{rz} = \frac{1}{2} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \end{aligned} \quad (6.42)$$

cluding viscosity, equation (2.2); multiply it by v_i and interchange dummy indices, to get an expression for the time-change of kinetic energy:

$$\rho \frac{D}{Dt} \frac{v^2}{2} = \sigma_{ik,i} v_k + \rho g_k v_k \quad (6.35)$$

Subtract this from the full energy equation (also written in Cartesian), and summetrize the tensor term (in the second step), to get two forms of the energy equation including viscosity:

$$\rho \frac{De}{Dt} = \sigma_{ik} v_{k,i} = \sigma_{ik} D_{ik}; \quad \rho \frac{De}{Dt} = -p \frac{dv_k}{dx_k} + \Sigma_{ik} D_{ik} \quad (6.36)$$

This brings back the (symmetrized) deformation tensor in Cartesian:

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \quad (6.37)$$

and introduces what I'll call the *dissipation function*: $\mathcal{D} = \Sigma_{ij} D_{ij}$. Referring back to (2.37) and (2.38), it's normal to assume $\nu_b = 2\nu/3$, which simplifies the stress and dissipation tensors. In this limit, still Cartesian, the energy equation becomes

$$\rho \frac{\partial e}{\partial t} + \rho v_i \frac{\partial e}{\partial x_i} + p \frac{\partial v_k}{\partial x_k} = \rho\nu \left(\frac{\partial v_k}{\partial x_k} \right)^2 \quad (6.38)$$

To close, I write out explicitly the dissipation functions for all 3 coordinate systems.

Spherical is even better (recall that θ is the polar angle and ϕ is the azimuthal angle):

$$\mathcal{D} = 2\rho\nu (D_{rr}^2 + D_{\theta\theta}^2 + D_{\phi\phi}^2 + 2D_{r\theta}^2 + 2D_{\theta\phi}^2 + 2D_{\phi r}^2) + \rho(\nu_b - 2\nu/3)(\nabla \cdot \mathbf{v})^2 \quad (6.43)$$

where

$$\begin{aligned} D_{rr} &= \frac{\partial v_r}{\partial r} ; & D_{\phi\phi} &= \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} ; & D_{\theta\theta} &= \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \\ D_{r\phi} &= \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \frac{v_\phi}{r} \right) ; & D_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \frac{v_\theta}{r} \right) \\ D_{\theta\phi} &= \frac{1}{2} \left(\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \frac{v_\phi}{\sin \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right) \end{aligned} \quad (6.44)$$

and this is the end of this chapter.