7. SIGNAL PROPAGATION

In this chapter we introduce a very important basic tool: the speed at which a disturbance propagates through a fluid. Because the disturbance we're considering is compressive – think of a locally overpressure region – one example is a sound wave. Thus the signal speed is called the *sound speed*: we'll see that it's given by

$$c_s^2 = \frac{\partial p}{\partial \rho} \tag{7.1}$$

First we'll derive this important speed, then look at how causality can dramatically change the nature of supersonic flows.

A. Sound Waves and the Signal Speed

This is worth two derivations; one physical, and another one more mathematical.

SOUND WAVES: A PHYSICAL APPROACH

We can also demonstrate that c_s is the speed of a travelling wave. Let some perturbation $(\delta\rho, \delta p, \delta T)$ be moving at some c_s. Ahead of the wave the fluid has $v = 0$; behind the wave the fluid has δv, in the same direction as the wave motion.

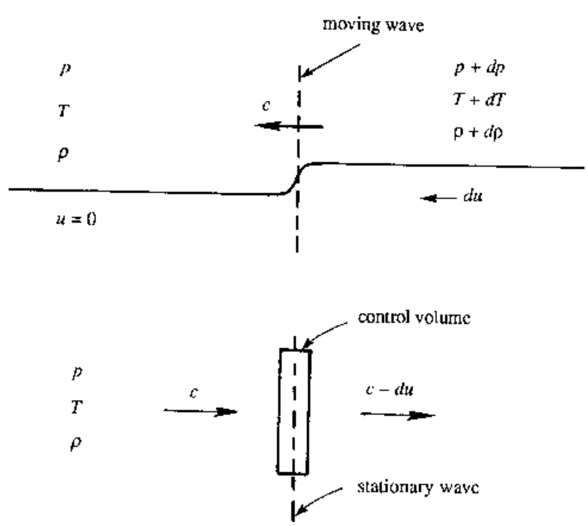

Figure 7.1. Propagation of a sound wave: (a) into a still fluid; (b) a stationary wave. From Kundu figure 15.1. Note, u here is the same as v in the text.

Mass conservation at the wave front, in a frame moving with the wave front, gives

$$\rho c_s = (\rho + \delta\rho)(c_s - \delta v)$$

and to lowest order small, this gives

$$\delta v \simeq c_s \frac{\delta\rho}{\rho} \tag{7.2}$$

Thus, $\delta v > 0$ if $\delta\rho > 0$; the passage of a compression wave leaves behind a fluid moving in the direction of the wave. Now, apply momentum balance: the net force on some control volume, from the pressure difference, equals the rate of change of momentum in that volume. That is,

$$p - (p + \delta p) = \rho c_s \left[(c_s - \delta v) - c_s\right]$$

so that, again to first order small, we have

$$\delta p \simeq \rho c_s \delta v \tag{7.3}$$

Combining (7.2) and (7.3), we get a condition on the wave speed (to allow mass and momentum balancs):

$$c_s^2 = \frac{\delta p}{\delta \rho} \tag{7.4}$$

Thus, we have one justification of our expression (7.1) for c_s.

SOUND WAVES: A FORMAL APPROACH

We can also use a more mathematical approach, deriving a formal wave equation by linearizing our two basic equations, continuity & momentum:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{7.5}$$

(assuming no body forces)

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p \tag{7.6}$$

Our question is, at what speed does a small, compressional disturbance in the gas travel? We start with a uniform, static unperturbed state, described by ρ_o, p_o and $\mathbf{v}_o = 0$. We add small perturbations, ρ_1, p_1 and $\mathbf{v}_1$. If we put these into the mass and momentum equations, (7.5) and (7.6), noting that the "o" terms are constant in space and in time, we get

$$\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_o + (\rho_o + \rho_1) \nabla \cdot \mathbf{v} = 0$$

and

$$(\rho_o + \rho_1) \frac{\partial \mathbf{v}_1}{\partial t} + (\rho_o + \rho_1) \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 = -\nabla p_1$$

In these equations, we now drop terms which are second order in the perturbed quantities, and we write $\nabla p = (\partial p / \partial \rho) \nabla \rho$. This gives us

$$\frac{\partial \rho_1}{\partial t} + \rho_o \nabla \cdot \mathbf{v}_1 = 0 \tag{7.7}$$

and

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \left(\frac{\partial p}{\partial \rho}\right) \nabla \rho_1 = 0 \tag{7.8}$$

Two approaches are possible here.

• **Method # 1**. This is more physically intuitive. From this pair of equations we can, for instance, eliminate $\mathbf{v}_1$, to get a second order DE in ρ_1:

$$\frac{\partial^2 \rho_1}{\partial t^2} = \left(\frac{\partial p}{\partial \rho}\right) \nabla^2 \rho_1 \tag{7.9}$$

This, of course, has travelling wave solutions: the disturbances travel at a speed given by

$$c_s = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} \tag{7.10}$$

This recovers our guess in (7.1).

• **Method # 2**. This is more formal, and a simple example of a general method which is the standard approach when the situation is more complex. (This will be useful in the homework for instance). Refer back to equations (7.7). They are linear in the perturbations, which allows us to use Fourier techniques. That is, we can consider one simple perturbation, say $\rho_1, \mathbf{v}_1 \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$; and we know that any arbitrary perturbation can be expressed as a sum of these $(\omega, \mathbf{k})$ waves.

With this, taking (7.1) as a definition (only that, for the moment), and dropping down to a 1D perturbation to simplify, equations (7.7) and (7.8) become

$$\begin{aligned} -i\omega\rho_1 + \rho_o ikv_1 &= 0 \\ -i\omega\rho_o v_1 + ikc_s^2\rho_1 &= 0 \end{aligned} \tag{7.11}$$

From these we immediately find the *dispersion relation*:

$$\omega^2 = c_s^2 k^2 \tag{7.12}$$

Thus, we have verified that we have non-dispersive waves propagating at speed c_s. This again verifies our guess in (7.1).

The derivative in (7.1) is usually taken assuming the disturbance is adiabatic, so that p/ρ^γ is constant, and $c_s^2 = \gamma p/\rho$. This corresponds, physically, to the heating and cooling times for the perturbation to be long compared to the wave travel time. An alternative choice which is sometimes used, is to consider isothermal perturbations: $T =$ constant – which is the limit in which the local cooling and heating times are short compared to the wave travel time. In this case, $c_s^2 = p/\rho$.

B. Why is the sound speed important?

I'm storing two important ideas in this section.

1. WHEN CAN WE ASSUME INCOMPRESSIBLE FLOW?

In the first five chapters, we assumed incompressible flow, and noted that this is a good approximation when the flow speed is much less than the sound speed. To start, we justify this assumption. We begin with the continuity equation:

$$\nabla \cdot (\rho \mathbf{v}) = 0 \tag{7.13}$$

This reduces to the incompressible condition, $\nabla \cdot \mathbf{v} = 0$, if (in 1D Cartesian for simplicity),

$$v\frac{\partial \rho}{\partial x} \ll \rho \frac{\partial v}{\partial x}\,; \qquad v\delta\rho \ll \rho\delta v \tag{7.14}$$

Now, the momentum/Euler equation becomes,

$$v\delta v \simeq \frac{1}{\rho}\delta p \tag{7.15}$$

If we now use the sound speed, from (7.1), we have $\delta p \simeq c_s^2 \delta\rho$, and thus

$$\frac{\delta\rho}{\rho} \simeq \frac{v^2}{c_s^2}\frac{\delta v}{v} \tag{7.16}$$

We thus find that the density changes are negligible – and thus we can work in the incompressible limit – when

$$\frac{v_2}{c_s^2} = \mathcal{M}^2 \ll 1 \tag{7.17}$$

Here, we have introduced the *Mach number*, $\mathcal{M} = v/c_s$.

2. THE IMPORTANCE OF CAUSALITY

We have just demonstrated that the sound speed is the speed at which information propagates; it is also critical to the dynamics of the flow.

There are important difference between subsonic and supersonic flows. Subsonic flows can be thought of as quasi-hydrostatic. That is, the flow field is strongly influenced by pressure gradients which are determined by conditions a long distance away (such as at boundaries).

Supersonic flows, however, are quasi-ballistic. This distinction is due to the fact that information travels at a finite speed, the speed of a simple sound wave. Both 1D and 3D cartoons, in Figures 7.1 and 7.2, can illustrate this point. Pressure gradients have only a limited range of influence, and conditions far away have little or no effect on a solution locally. We'll see that supersonic flows can (and usually do) contain discontinuous jumps in the flow properties (shocks). They can violate our subsonic intuition, for instance a supersonic flow in a *diverging* channel will *accelerate* (as we'll see below).

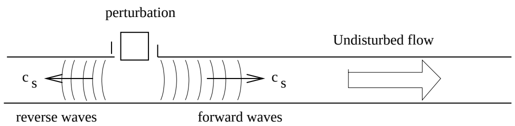

Figure 7.2. Physical illustration of simple waves. The information that the flow has been "whacked" at point a, propagates by simple sound waves, moving at speed c_s relative to the fluid in the pipe. Following Thompson figure 8.6

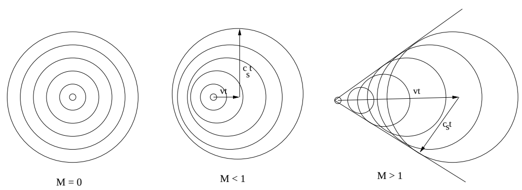

Figure 7.3. Mach's construction for the propagation of a disturbance. Consider a point source of sound (Thompson suggests a bumblebee) in a moving medium. If the source and flow are stationary, the sound propagates spherically from the source. If the source/flow are moving subsonically. the motion only distorts the spherical wavefronts. If, however, the motion is supersonic. all disturbances are confined to a *Mach cone*; an observer located outside of this cone does not receive any information about the bee. The opening angle of the cone is called the *Mach angle*: $\sin\mu = 1/\mathcal{M}$.

Some authors use the following classifications, based on the Mach number $\mathcal{M} = v/c_s$ (I follow Kundu, for instance):

• *incompressible*: $\mathcal{M} \lesssim 0.3$ everywhere; can ignore density variations due to pressure changes.

• *subsonic*: $0.3 \lesssim \mathcal{M} \lesssim 1$ everywhere. Need to be careful with density fluctuation (now above the 10% level), but no shock waves in the flow.

• *transonic*: $0.8 \lesssim \mathcal{M} \lesssim 1.2$, say; $\mathcal{M}$ is "around unity". Shock waves appear, increasing drag. These are the hardest flows to analyze analytically, as the nonlinear terms in the governing equations are important, but the simplifying effects of $\mathcal{M} > \infty$ flow can't be used yet.

• *supersonic*: $1 \lesssim \mathcal{M} \lesssim 3$, typically. Shock waves are usually present. Analysis is easier because information propagates along well-defined directions in (x, t) space, called *characteristics*.

• *hypersonic*: $\mathcal{M} \lesssim 3$, say. Qualitatively the same as supersonic flow, but some interesting new effects – such as strong heating and ionization of boundary layers – come into play. Shocks can be analyzed in the strong-shock limit (chapter 9).

C. Weak Waves and Causality

How can we use the signal speed – the sound speed – to understand a flow? One way to approach this, following Currie, is to consider "weak waves". Specify to a 1D system, and let c_o be the undisturbed value of c_s. Equations (7.7) and (7.8) become

$$\frac{\partial \rho_1}{\partial t} + \rho_o \frac{\partial v_1}{\partial x} = 0 \ \ ; \quad \rho_o \frac{\partial v_1}{\partial t} + c_s^2 \frac{\partial \rho_1}{\partial x} = 0$$

In the previous derivation, c_s can be a function of density, and thus can vary within the wave. Here, we simplify by taking c_s to be a constant, c_o. But now: because ρ_o is assumed constant, and because $v_o = 0$, we can write

$$\frac{\partial \rho}{\partial t} + \rho_o \frac{\partial v}{\partial x} = 0 \ \ ; \quad \rho_o \frac{\partial v}{\partial t} + c_o^2 \frac{\partial \rho}{\partial x} = 0 \qquad (7.18)$$

Compare these to the originals, (7.7) and (7.8): we have quietly removed the bothersome, nonlinear terms. This is valid only as long as we continue to assume a small perturbation – a "weak wave". We can now divide the first of (7.18) by ρ_o, the second by $\rho_o c_o$, and add and subtract, to get

$$\frac{\partial}{\partial t}\left(\frac{v}{c_o} + \frac{\rho}{\rho_o}\right) + c_o \frac{\partial}{\partial x}\left(\frac{v}{c_o} + \frac{\rho}{\rho_o}\right) = 0 \ \ ;$$
$$\frac{\partial}{\partial t}\left(\frac{v}{c_o} - \frac{\rho}{\rho_o}\right) - c_o \frac{\partial}{\partial x}\left(\frac{v}{c_o} - \frac{\rho}{\rho_o}\right) = 0 \qquad (7.19)$$

These now have the form of a total (Lagrangian) derivative (compare equation 1.5). Thus, we have an important result:

$$\left(\frac{v}{c_o} + \frac{\rho}{\rho_o}\right) = \text{constant} \quad \text{on} \quad x - c_o t = \text{constant} \qquad (7.20)$$

and

$$\left(\frac{v}{c_o} - \frac{\rho}{\rho_o}\right) = \text{constant} \quad \text{on} \quad x + c_o t = \text{constant} \qquad (7.21)$$

An alternate form of (7.20 and 7.21) can be found with a bit of algebra. Using (7.7) and $\rho_1 \ll \rho_o$, it's easy to express ρ/ρ_o in terms of p/p_o and γ, to write

$$J^+ = \left(\frac{v}{c_o} + \frac{1}{\gamma}\frac{p}{p_o}\right) = \text{constant} \qquad (7.22)$$
$$\text{on} \quad x - c_o t = \text{constant}$$

and

$$J^- = \left(\frac{v}{c_o} - \frac{1}{\gamma}\frac{p}{p_o} \right) = \text{constant} \qquad (7.23)$$
$$\text{on} \quad x + c_o t = \text{constant}$$

The interpretation is simple for (7.20) or (7.22). The lines $x = \pm c_o t$ are the loci of forward and backward propagating sound waves. Our results say that the quantities on the left of equations (7.21) and (7.23) are constant along these trajectories.

D. Two examples of simple waves

Two standard examples show how this analysis can be used.

1. SHOCK TUBE

Consider a gas confied to a 1D tube. A diaphragm at $x = 0$ separates high-pressure ($p_1, x < 0$) gas from low-pressure ($p_0, x > 0$) gas. At $t = 0$ the diaphragm breaks, and the high-pressure gas begins to expand into the low-pressure gas. The information that this has happened can only travel at the sound speed, c_o. Thus, a compression wave will move to the right, and an expansion wave to the left, both at c_o. The problem: what is the velocity and pressure of the gas everywhere, as a function of time?

Figure 7.4 shows an (x, t) diagram for this system, with the two wave paths as solid lines. Gas ahead of the forward wave must be undisturbed: it has $p = p_o, v = 0$. Similarly, gas behind the reverse wave has $p = p_1, v = 0$. What of the middle region? Consider an observation point $P(x_p, t_p)$. Two wave lines intersect P: one forward wave which originates from $(x < x_p, t = 0)$, and a reverse wave which originates from $(x > x_p, t = 0)$. But now: we know $v = 0$ everywhere at $t = 0$, so we can evaluate the constants in (7.22, 7.23) for each of these wave lines. For the reverse one, we have $J^- = -1/\gamma$; and for the forward one, we have $J^+ = p_1/\gamma p_o$ (verify this for yourself!). Thus, at point P we know that

$$\frac{v}{c_o} + \frac{1}{\gamma}\frac{p}{p_o} = \frac{1}{\gamma}\frac{p_1}{p_o} \quad \text{and} \quad \frac{v}{c_o} - \frac{1}{\gamma}\frac{p}{p_o} = -\frac{1}{\gamma}$$

These contain enough information to solve the system. We find that

$$v = \frac{c_o}{2\gamma}\frac{p_1 - p_o}{p_o} \;\; ; \quad p = \frac{p_1 + p_o}{2} \qquad (7.24)$$

Thus: the region inbetween the two waves has a uniform pressure, equal to the mean of p_1 and p_o; and it moves to the right at a uniform speed, proportional to the initial pressure difference.

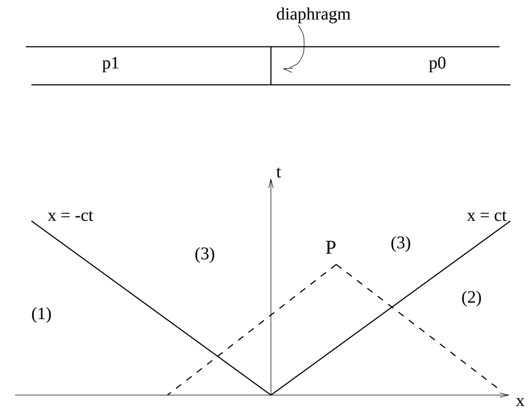

Figure 7.4. Illustrating the geometry of the ruptured-diaphragm problem, in the x,t plane. The two solid lines are the forward and reverse waves which start at the origin; the two dotted lines are the waves which intersect the observation point P. Following Currie Figure 11.2.

2. PISTON PROBLEM

Consider another 1D tube filled with gas; at $t = 0$ a piston is moved into the gas, at velocity U. How does the gas respond?

In Figure 7.5 we show this problem in the (x, t) plane. Once again, we expect the gas to be undisturbed (pressure p_o, zero velocity) ahead of the forward-wave which starts from the piston at $t = 0$. Also again, consider an observation point $P(x, t)$. It connects the the forward x-axis by a reverse wave; we can evaluate the constant J^-, because we know $v = 0, p = p_o$ at the x-axis. The point P also connects to the *piston* by a forward wave. We cannot evaluate $J++$ yet, however; we know the velocity $v_p = U$, but not the pressure, p_p, at the piston. We thus need a third wave, another reverse one which connects the piston to the x-axis. We thus have three algebraic equations:

$$\begin{aligned} \frac{U}{c_o} - \frac{1}{\gamma}\frac{p_p}{p_o} &= -\frac{1}{\gamma} \\ \frac{v}{c_o} + \frac{1}{\gamma}\frac{p}{p_o} &= \frac{U}{c_o} + \frac{1}{\gamma}\frac{p_p}{p_o} \\ \frac{v}{c_o} - \frac{1}{\gamma}\frac{p}{p_o} &= -\frac{1}{\gamma} \end{aligned} \qquad (7.25)$$

This now solves the system: we can find p_p, and then solve for conditions in the gas between the piston and the forward wave:

$$v = U \;\; ; \quad \frac{p}{p_o} = \gamma\frac{U}{c_o} + 1 \qquad (7.26)$$

3. WAVES AT BOUNDARIES

Finally, a comment about wave reflection at boundaries. Waves reflect in like manner off solid boundaries. This

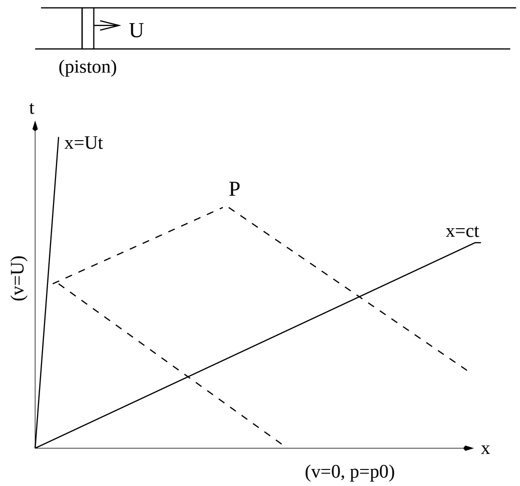

Figure 7.5. Illustrating the geometry of the piston problem, in the x,t plane. The two solid lines are the locus of the piston (assumed to be close to vertical; $U \ll c_o$), and the forward wave starting from the origin. The dotted lines are the two waves which intersect the observation point P, and the third (reverse) wave used to connect the piston to the $x > 0$ axis. Following Currie Figure 11.5.

means that at a solid wall, a compression wave reflects as a compression wave, and an expansion wave reflects as an expansion wave. Free boundaries are different; waves reflect in an unlike manner. That means that expansion waves reflect as compression waves, and vice versa. The reason is that along a solid boundary, the flow direction is the boundary condition, while off a free boundary the pressure is the boundary condition.

References

Good references here include Thompson, Currie, Kundu and Faber.