8. ONE-DIMENSIONAL STEADY FLOW

We found in Chapter 7 that the sound speed, c_s, is the speed at which a signal propagates in a fluid. Because of this, there are important differences between subsonic and supersonic flow.

Although most flows are (at least!) two-dimensional, we can learn a lot from the simpler case of 1D flows. These are often called *channel flows*; think of a firehose, the Alaska pipeline, or a wind tunnel. If you're an astrophysicist, you can think about jets (from protostars, or accretion disks around black holes). Another astrophysical application is to spherically symmetric flows (such as the solar wind), which are effectively one-dimensional.

A. Two-point Connections in Steady Flow

One approach, especially in the engineering-type literature, is to connect flow properties "here" to those "there".

It's useful in this game to remember the **Adiabatic Relations**. Steady flows are often assumed to be adiabatic, with a specific heat γ. This gives us useful scaling laws:

$$\frac{T_f}{T} = \left(\frac{c_{sf}}{c_s}\right)^2 = \left(\frac{p_f}{p}\right)^{(\gamma-1)/\gamma} = \left(\frac{\rho_f}{\rho}\right)^{\gamma-1} \qquad (8.1)$$

In the above set of expressions, the subscript "f" can refer to any fiducial point ("there") at which we know T, c_s, p or ρ; the unsubscripted quantities then refer to "here".

To be specific, now, consider 1D flow in a channel, or pipe, of cross-section A. To sound more formal, let "here" be region 1, and "there" be region 2 (or vice versa, of course). We will want to apply two conservation laws.

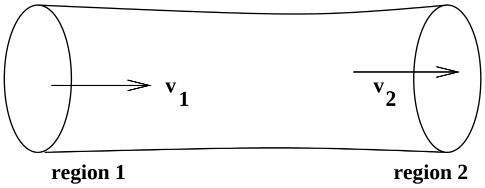


Figure 8.1. A simple 1D "channel", which has cross-section area A. We want to connect conditions in region 1 (ρ_1, T_1, p_1, v_1) to those in region 2 (ρ_2, T_2, p_2, v_2).

Mass conservation has

$$\rho v A = \text{constant} \qquad (8.2)$$

In our current application, this becomes

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \qquad (8.3)$$

Energy conservation, otherwise known as Bernoulli, has

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{1}{2}v^2 = h + \frac{1}{2}v^2 = \text{constant} \qquad (8.4)$$

(recall $h = e + p/\rho$ is the enthalpy). This becomes

$$h_1 + \frac{1}{2}v_1^2 = h_2 + \frac{1}{2}v_2^2 \qquad (8.5)$$

Energy conservation with heat addition sometimes appears in 1D problems. In this case, Bernoulli is modified as

$$h_1 + \frac{1}{2}v_1^2 + q_{12} = h_2 + \frac{1}{2}v_2^2 \qquad (8.6)$$

if some quantity q_{12} of heat (per gram) is added between points 1 and 2. This could describe, for instance, to a turbojet, in which ignition of a (hopefull tiny!) amount of fuel adds heat to the flow, and accelerates it (if the flow is subsonic – more on that in the homework).

Two specific **reference points** in the flow are sometimes useful.

The *stagnation point* is just that, a point on a streamline where the velocity goes to zero. The subscript "o" may be used here. One refers, for instance, to h_o, the "stagnation enthalpy". This is often connected to a "reservoir" – picture a large tub of gas, from which a flow emerges through a nozzle.

Some useful relations here start from

$$c_s^2 + \frac{\gamma-1}{2}v^2 = c_{so}^2 \qquad (8.7)$$

(this is just Bernoulli again, right?), which can also be written,

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2}\mathcal{M}^2 \qquad (8.8)$$

From this, the ratios ρ/ρ_o, p/p_o, come simply from the adiabatic relations (eqns 8.1).

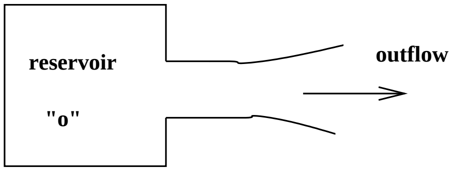


Figure 8.2. A 1D flow from a "reservoir" out into the universe. The gas in the reservoir is assumed to have zero velocity, and carries subscript o: its conditions are ρ_o, p_o, T_o.

The *sonic point* is another useful reference – the point where $v = c_s$. The subscripts "*" or "s" are used here. The sonic and stagnation points can be connected simply:

$$\frac{T_*}{T_o} = \frac{2}{\gamma+1}\,; \qquad \frac{p_*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{(\gamma/\gamma-1)}\,; \qquad \frac{\rho_*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1)} \tag{8.9}$$

Mass conservation can be used to derive another useful relation, between the sonic point and any general point in the flow with area A and Mach number $\mathcal{M}$:

$$\left(\frac{A}{A_*}\right)^2 = \frac{1}{\mathcal{M}^2}\left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\mathcal{M}^2\right)^{\frac{\gamma+1}{\gamma-1}} \tag{8.10}$$

B. One-dimensional channel flow

Another approach is to consider the structure of the flow throught the channel. In particular, consider flow in a channel of area A; work in the limit where the flow is 1D, that is ignore cross-channel variation. We're particularly interested in the effect that variations in A have on the flow properties.

The steady continuity and momentum equations are

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\frac{dA}{dx} = 0 \tag{8.11}$$

and

$$\rho v\frac{dv}{dx} + \frac{dp}{dx} = 0 \tag{8.12}$$

Using $c_s^2 = dp/d\rho$, these combine as

$$\left(\mathcal{M}^2 - 1\right)\frac{1}{v}\frac{dv}{dx} = \frac{1}{A}\frac{dA}{dx} \tag{8.13}$$

which is the basic, governing equation.

1. NOZZLES AND DIFFUSERS

This leads to a striking result: the behavior of the flow in a channel depends on whether it is subsonic or supersonic. In particular, compare the flows in converging and diverging channels:

- At subsonic speeds ($\mathcal{M} < 1$), a decrease of area increases the flow speed. A subsonic nozzle (which accelerates the flow) must therefore have a convergeant profile, and a subsonic diffuser (which decelerates the flow) must have a divergent profile. Compressibility does not change this qualitative behavior.

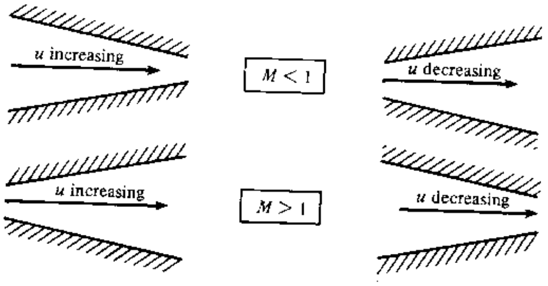
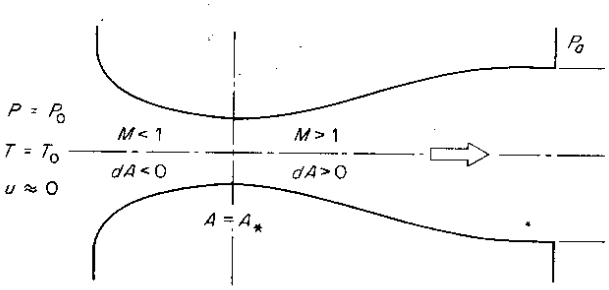


Figure 8.3. Nozzles and diffusers in subsonic and supersonic regimes. A converging nozzle accelerates subsonic flow, and decelerates supersonic flow. A diverging nozzle decelerates subsonic flow, and accelerates supersonic flow. From Anderson Figure 5.3.

- At supersonic speeds, however, ($\mathcal{M} > 1$), a decrease of area leads to a *decrease* of speed, and conversely. Thus, a supersonic nozzle must be divergeant; and a supersonic diffuser myst be convergeant. The reason for this can be noted from the force equation, which can be written $d\rho/\rho = -\mathcal{M}^2 dv/v$ so that, for supersonic flows, the density decreases faster than the velocity increases; so the area in an accelerating flow must increase, in order to keep the product $A\rho v$ constant.

2. A SMOOTH TRANSITION?

What about connecting these? It is possible to accelerate a flow smoothly through the sound speed, by a proper combination of converging and diverging channels. Figure 8.2 shows an example of this. By considering (8.13), we can see that such flow must reach $\mathcal{M} = 1$ at the throat (where $dA/dx = 0$). Such a configuration is called a supersonic nozzle, a converging-diverging nozzle, or a de Laval nozzle.[1]



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[1] The last name is for Carl de Laval, a Swedish inventor who developed such a nozzle, in 1883; he was best known for developing a high-speed turbine for driving a cream separator. Astronomers may be familiar with the name as a model for the formation of radio jets by pressure confinement in active galactic nuclei.

Figure 8.4. Transonic flow in a a convergent-divergent nozzle. From Thompson Figure 6.3

We must note, however, that $\mathcal{M} = 1$ is not necessarily reached at the throat. Other combinations of initial/boundary conditions allow other flows. Equation (8.13) also allows smooth solutions where the velocity v is an extremum: everywhere subsonic flow can have a maximum of v at the throat, and everywhere supersonic flow can have a minimum. Figure 8.5 illustrates this.

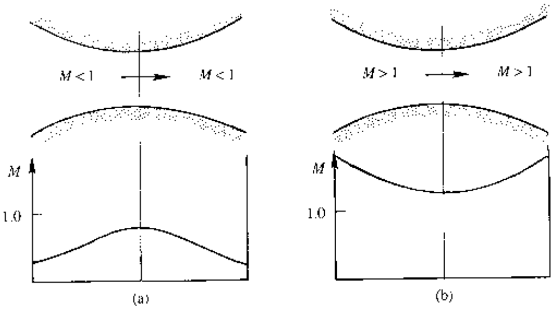

Figure 8.5. More convergent-divergent nozzles, in which the velocity does not reach $\mathcal{M} = 1$ at the throat. From Kundu figure 15.7

3. NORMAL SHOCKS

Most flows that go between subsonic and supersonic aren't so smooth, however (in fact it takes very special conditions to achieve the smooth, de Laval transition of Figure 8.4). More typically, mismatched variations in channel area and/or external pressure lead to shocks in the flow. Chapter 9 is devoted to shock properties; but we also need some short discussion here.

In brief, a *shock* is a discontinuous transition in the flow. If you force a supersonic flow to change suddenly (compared to the sound travel time), a shock forms at the right place and strength to accomodate the change.

Here we'll look at *normal shocks*, which means shocks normal to the flow.[2] Put yourself in a frame in which the shock is at rest; let "1" be conditions upstream (flow coming into the shock) and "2" be conditions downstream.

We apply mass and momentum conservation across the shock. These two conditions become

$$\rho_1 v_1 = \rho_2 v_2 \tag{8.14}$$

and

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{8.15}$$

We also apply energy conservation, assuming an adiabatic shock:

$$\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 \tag{8.16}$$

Now, these equations can be used to express the three post-shock quantities, ρ_2, v_2 and p_2, in terms of their pre-shock counterparts. Working through the algebra (see chapter 9) gives the *normal shock jump conditions*:

$$\begin{aligned} \frac{\rho_1}{\rho_2} &= \frac{\gamma-1}{\gamma+1} + \frac{1}{\mathcal{M}^2}\frac{2}{\gamma+1} \\ \frac{p_2}{p_1} &= \frac{2\gamma\mathcal{M}^2 - (\gamma-1)}{\gamma+1} \\ \frac{v_2}{v_1} &= \frac{\gamma-1}{\gamma+1} + \frac{1}{\mathcal{M}^2}\frac{2}{\gamma+1} \end{aligned} \tag{8.17}$$

Once you have these, the jumps in T and c_S can also be derived from the usual adiabatic and ideal gas relations.

In a nutshell: a shock decelerates the flow ($v_2 < v_1$). The density must increase, due to mass conservation ($\rho_2 > \rho_1$). The "lost" kinetic energy goes to heat: $T_2 > T_1$ and $p_2 > p_1$.

And a caveat: equations (8.17) hold for the *rest frame* of the shock. In many applications,however, the shock moves through the fluid. If this is the case, you'll need to remember to do a Galilean transform to the shock-rest frame before applying the jump conditions.

4. 1D FLOWS WITH INTERNAL SHOCKS

Now, let's return to 1D flows, and consider what happens if smooth flow is not reached.

The pressure jump maintained across the nozzle can determine the flow possibilities. From (8.12), the sign of dp is opposite to the sign of dv; accelerating flow has a pressure drop, and vice versa. Consider, say, a nozzle connecting an input region of pressure p_o, and an exit at p_e. What happens? (I follow Anderson in this discussion; refer to Figure 8.6.)

(a) For $p_e = p_o$, there is no flow of course. Let p_e drop a bit (say to p_a in the figure): the flow is completely subsonic, with highest $\mathcal{M}$ and lowest p at the throat.

(b) At some specific value of p_e, the flow reaches sonic at the throat. It remains subsonic everywhere else, however (p_b is not small enough to allow a smooth sonic transition); this is called *choked* flow.

(c) As the exit pressure is further reduced, a region of supersonic flow occurs downstream of the throat. Because it can't continue so all the way to the exit, a

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[2] No, there are no "abnormal" shocks, sorry.

normal shock forms in the diverging nozzle; the flow slows down to subsonic at the shock.

(d) For some specific value of p_e, the shock is located exactly at the exit. The fully isentropic, subsonic-supersonic flow pattern now exists throughout the entire duct, except at the exit.

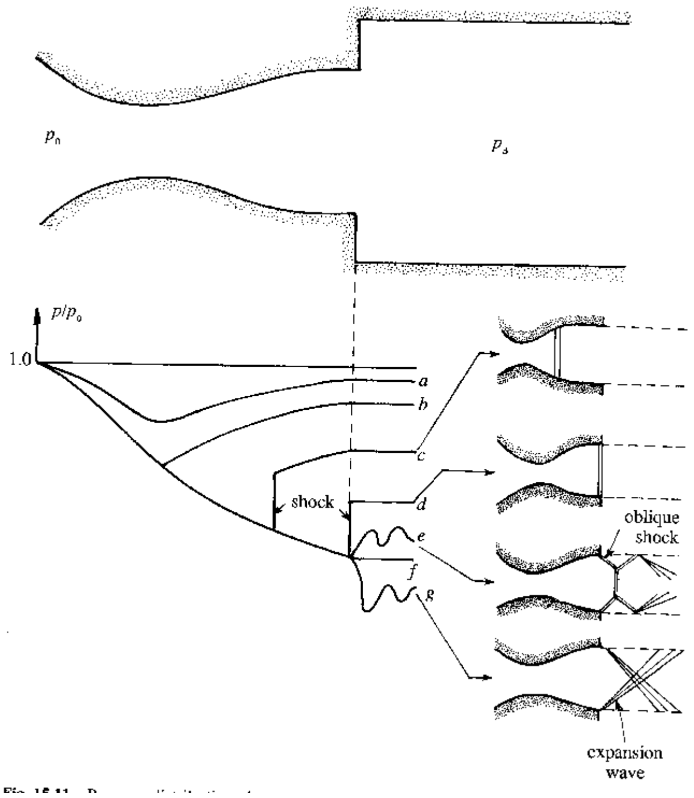

Figure 8.6. The effect of the downstream pressure on a convergent-divergent nozzle. See text for details. From Kundu figure 15.11

(e) As p_e is reduced further, the normal shock is replaced by oblique shocks emanating from the edge of the nozzle. This is *overexpanded* nozzle flow.

(f) Finally, there is one value of p_e that is just right, exactly matching that required by the isentropic flow solution. No shocks of any kind exist here, the flow exits quietly into the post-nozzle region.

(g) At still lower values of p_e, expansion waves will emanate from thet edge of the nozzle. This is an *underexpanded* nozzle flow.

C. Spherical stellar wind flow

A different example of one-dimensional "channel" flow is spherical outflow from a central mass: a stellar wind. Consider a spherical expansion, driven against gravity by a steady mass and energy source at the origin. In this case, the gravity plays the role of the area in channel flow. In particular, a smooth transition from subsonic to supersonic flow is possible if the gas stays hot enough (extended heating sources are required).

The basic solution is due to Parker. Consider a steady, spherical outflow. Mass conservation in this case is $\rho v r^2$ =constant; or,

$$\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{v}\frac{dv}{dr} + \frac{2}{r} = 0 \tag{8.18}$$

while the momentum equation becomes in this case (noting that gravity from the central star is important),

$$\rho v \frac{dv}{dr} + \frac{dp}{dr} = -\rho \frac{GM}{r^2} \tag{8.19}$$

Writing $dp/dr = c_s^2 d\rho/dr$, these two equations combine to give the basic wind equation,

$$\left(v - \frac{c_s^2}{v}\right)\frac{dv}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2} \tag{8.20}$$

This does not have analytic solutions over the whole range of r. It must be solved numerically; examples are shown in Figure 8.6. However, as with our analysis of channel flow, we can learn a lot by simple inspection of (8.20).

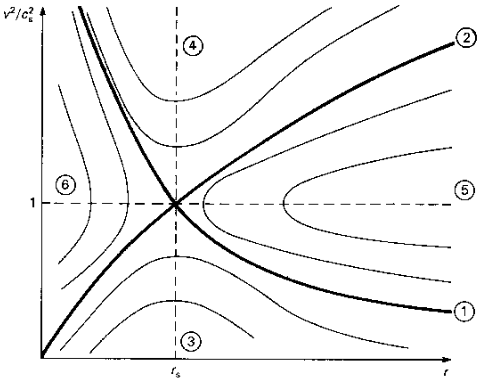

Figure 8.7. Possible solutions of the wind equation, (8.20). Initial or boundary conditions determine which of these solutions will exist in a particular system. The heavy lines show the two interesting solutions, which pass through a sonic point (at r_s) and connect smoothly between $r \to 0$ and $r \to \infty$. The outflow solution represents a stellar wind, and the inflow solution represents steady, spherical accretion. From Frank, King & Rraine, Figure 2.1.

(i) The left hand side contains a zero, at $v^2 = c_s^2$. If we want to consider well-behaved flows, that is to say those in which the derivative dv/dr does not blow up, then the right hand side of (8.20) must go to zero at the same point. This defines the condition that must be met at the sonic point:

$$v^2 = c_s^2 \quad \text{at} \quad r = r_s = \frac{GM}{2c_s^2} \tag{8.21}$$

Whether or not a particular flow satisfies this condition depends on the starting conditions, such as with what velocity and temperature it left the stellar surface, and also what the boundary conditions at large distances are. If it does not start in such a way to satisfy this condition, it either stays subsonic (corresponding to finite pressure at infinity), or cannot establish a steady flow.

(ii) The solution beyond the sonic point depends on the temperature structure of the wind. The only solutions with $dv/dr > 0$ for $r > r_s$ are those for which $c_s^2(r)$ drops off more slowly than $1/r$; it is only these for which the right-hand side stays positive. In the case of an isothermal wind, with $c_s^2 =$ constant, (8.20) can be solved in the limit $r >> r_s$:

$$v^2(r) \simeq 4c_s^2 \ln r = \text{constant} \tag{8.22}$$

Thus, the wind will be supersonic, by a factor of a few, as $r \to \infty$. The question of how the solar wind manages to stay nearly isothermal is not solved; it is probably due to energy transport by some sort of waves (MHD or plasma waves, for instance) which are generated in the photosphere and damped somewhere far out in the wind.

(iii) Inside the sonic point, the gravity term will dominate the right hand side of (8.20). Thus, solutions with $dv/dr > 0$, and $v^2 < c_s^2$, will obey

$$v\frac{dv}{dr} \simeq -\frac{GM}{r^2}$$

This equation looks as if gravity is driving the wind out! This unlikely-looking result comes from the fact that the flow is nearly subsonic in this region; therefore, the dp/dr term in (8.19) – which actually drives the wind out – is nearly equal to the gravity term.

What happens at the outer edge of such a wind? Remember that the wind is not flowing into vacuum, but rather into some external medium (call it the ISM, for interstellar medium: this theory has been developed to describe the solar wind). The pressure in the wind is dropping with radius (because the density is dropping, right?). When the wind pressure is close to the ambient pressure, the wind must slow down. In fact, we expect a two-shock structure. At this outer boundary of the wind, we expect some sort of shock transition, since the wind is supersonic. Past this shock, the hot, shocked wind-gas will expand into the ISM (at about its own sound speed, to start); as long as this expansion is supersonic relative to the ISM, the expanding hot gas will drive a "snowplowed" shell of ISM, and a second shock, out into the ISM.

A cartoon of this region, at some point in time, would be that in Figure 3.2. Let region "a" be the wind; S_1 be the inner shock; region "b" be the wind-gas which has been through the shock; C be the contact surface between the wind and the ISM; region "c" be the shocked ISM; and S_2 be the outer shock (moving into the ISM). We expect S_1 to be an adiabatic shock (since the wind is probably hot and low density, and thus will have a long cooling time); region "b" will contain hot, shocked wind, with $T_b \sim \frac{3}{16}\frac{mv_{wind}^2}{k_B} \sim \text{several} \times 10^7\text{K}$ (noting that $m = \frac{1}{2}m_p$ is the mean mass per particle if region "b" is fully ionized). The outer shock will probably be isothermal, since the ISM is denser and cooler than the wind. Thus, the shocked ISM will be in a thin shell, containing all of the original ISM that lay between S_2 and the star.

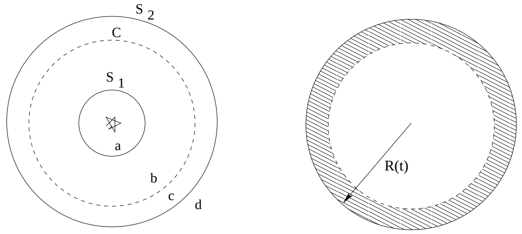

Figure 8.8. Cartoon of the structure of a stellar wind, and its interaction with the ISM. Left: the shock structure within the wind. Right: The outer shell of dense, snowplowed ISM. From Dyson & Williams figures 7.3 and 7.4.

We can't say anything about the details of the shock transitions until we know more about shocks ... and that's next.

References

The basic 1D flow material can be found in several places; I've followed Thompson, Kundu and Anderson, mostly. The more specialized spherical-wind applications are found in, for instance, Frank, King & Raine, *Accretion Power in Astrophysics*; and Dyson & Williams, *Physics of the Interstellar Medium*.