9. SHOCKS IN FLUID FLOW

Next, assume we have formed a shock, and consider how it affects the fluid moving through it. We idealize the shock as an infintesimally thin discontinuity in the flow. This allows us to use conservation laws to determine the "jumps" in macroscopic quantities across the shock (but of course it does not allow us to say anything about the fluid flow within the shock).

A. Jump conditions

To start, recall our three basic equations – conservation of mass, momentum, energy – written in conservative form. Here, we ignore body forces, viscosity, and heating/cooling losses. The basic equations, then, are:

$$\frac{\partial}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{9.1}$$

(from 1.4);

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + \mathbf{P}) = 0 \tag{9.2}$$

from (1.13), recalling $\mathbf{P}$ is the pressure tensor; and

$$\frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho v^2\right) + \nabla \cdot \left[\mathbf{v}\left(\rho e + p + \frac{1}{2}\rho v^2\right)\right] = 0 \tag{9.3}$$

(from 6.28). Comment: we are ignoring viscosity (it makes this job *so* much easier ...). You remember that viscosity depends on the second derivative of the velocity field. It follows that viscosity is critical to the *internal* structure of the shock; but not to the jump conditions.

We use these conservation laws to find jump conditions. We work in a frame moving with the shock, and assume a steady flow in that frame, so $\partial/\partial t = 0$. (Other frames require a Galilean transform to this *shock frame*). General notation: the incoming (upstream) quantities are labelled "1", the outgoing (downstream) quantites are labelled "2", and $[\![A]\!] = A_2 - A_1$ is the jump in A across a boundary. Let $\hat{\mathbf{n}}$ be the vector normal to the boundary, $\hat{\mathbf{t}}$ be a unit vector tangential to the boundary, and work in a frame in which the shock is stationary. Our conservation laws become, then,

- Mass flux:

$$[\![\rho \mathbf{v} \cdot \hat{\mathbf{n}}]\!] = 0 \tag{9.4}$$

which describes continuity of mass flux.

- Momentum flux (note $\mathbf{P}$ is a trace-only tensor if we ignore viscosity:)

$$[\![\rho \mathbf{v}(\mathbf{v} \cdot \hat{\mathbf{n}}) + p\hat{\mathbf{n}}]\!] = 0 \tag{9.5}$$

is the basic momentum flux. One useful consequence of this (resembling Bernoulli's equation) comes from dotting (again) it with $\hat{\mathbf{n}}$:

$$[\![p + \rho(\mathbf{v} \cdot \hat{\mathbf{n}})^2]\!] = 0 \tag{9.6}$$

We can also dot (9.5) with $\hat{\mathbf{t}}$, and use (9.4), to show

$$[\![(\mathbf{v} \cdot \hat{\mathbf{t}})]\!] = 0 \tag{9.7}$$

that is, the component of velocity in the shock plane does not change.

- Energy flux (recall $e + p = (\gamma/\gamma - 1)p$:)

$$\left[\!\!\left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1}p\right)\mathbf{v} \cdot \hat{\mathbf{n}}\right]\!\!\right] = 0 \tag{9.8}$$

B. Normal shocks

We first consider normal shocks, for which $\mathbf{v} \parallel \hat{\mathbf{n}}$. In this case, the system above reduces to the following. Continuity gives

$$\rho_1 v_1 = \rho_2 v_2 \tag{9.9}$$

Momentum conservation gives

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{9.10}$$

The energy jump condition, factoring out ρv from each side, can be written in the case of an adiabatic shock,

$$\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 \tag{9.11}$$

Now, these equations can be used to express the three post-shock quantities, ρ_2, v_2 and p_2, in terms of their pre-shock counterparts. One way to do this is to take the density jump as the basic variable: $X = \rho_2/\rho_1$. One can then eliminate the ratios p_2/p_1, and c_{s2}/c_{s1}, from the equations, deriving a quadratic in X:

$$X^2\left[2 + \mathcal{M}^2(\gamma - 1)\right] - 2X\left(1 + \mathcal{M}^2\gamma\right) + \mathcal{M}^2(\gamma + 1) = 0$$

and the positive solution of this gives the interesting solution. Writing out the ratios, we get the jump conditions:

$$\begin{aligned}
\frac{\rho_1}{\rho_2} &= \frac{\gamma - 1}{\gamma + 1} + \frac{1}{\mathcal{M}^2}\frac{2}{\gamma + 1} \\
\frac{p_2}{p_1} &= \frac{2\gamma\mathcal{M}^2 - (\gamma - 1)}{\gamma + 1} \\
\frac{v_2}{v_1} &= \frac{\gamma - 1}{\gamma + 1} + \frac{1}{\mathcal{M}^2}\frac{2}{\gamma + 1}
\end{aligned} \tag{9.12}$$

(Jumps in T and c_S can also be derived). We can also find a condition for the change in the Mach number:

$$\mathcal{M}_2^2 = \frac{\mathcal{M}_1^2 + 2/(\gamma-1)}{2\gamma/(\gamma-1)\mathcal{M}_1^2 - 1} \tag{9.13}$$

(In this last expression I have explicitly put ’1” and “2” subscripts on the Mach number; when omitted, $\mathcal{M}$ is generally taken to refer to the upstream value.) These expressions (9.12) & (9.13) are simple enough to evaluate. Nonetheless, most fluid books in the engineering tradition present shock tables – tabular forms of the solutions as functions of $\mathcal{M}$.

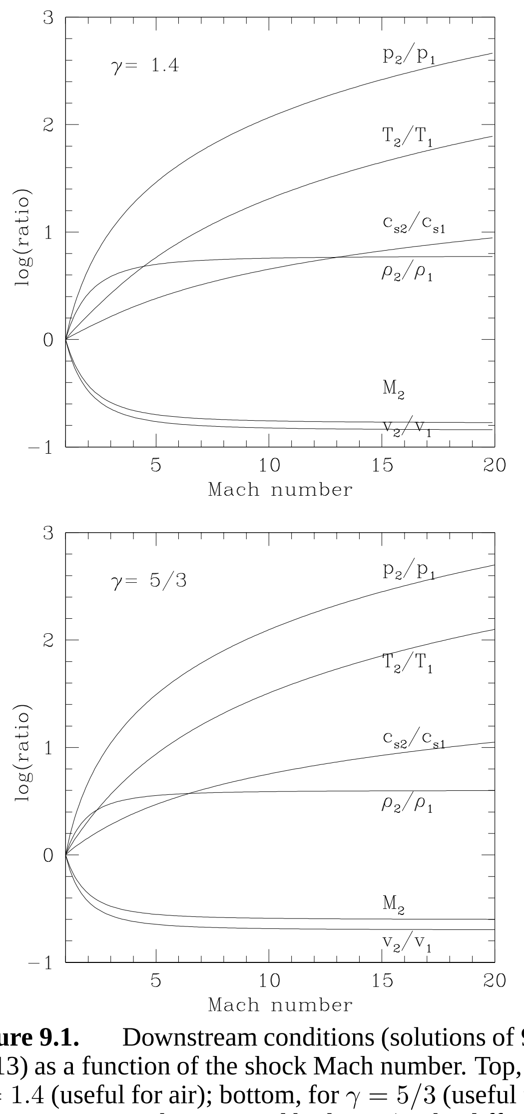

Figure 9.1. Downstream conditions (solutions of 9.12, 9.13) as a function of the shock Mach number. Top, for $\gamma = 1.4$ (useful for air); bottom, for $\gamma = 5/3$ (useful for a monatomic gas, such as ionized hydrogen). The differences in γ are apparent in the

STRONG SHOCK LIMIT, NORMAL SHOCKS

When $\mathcal{M} >> 1$, these equations simplify, to

$$\frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1}\,; \qquad \frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}^2}{\gamma+1}\,;$$
$$\frac{v_2}{v_1} = \frac{\gamma-1}{\gamma+1}\,; \qquad \mathcal{M}_2^2 = \frac{\gamma-1}{2\gamma} \tag{9.14}$$

In particular, if we consider an ideal gas with $\gamma = 5/3$, and let $\mathcal{M}$ become large, we find $\rho_2/\rho_1 = v_1/v_2 = 4$. This is the *strong shock limit*. Also in this limit, the temperature jump is $T_2/T_1 = 5\mathcal{M}^2/16$, giving $T_2 = 3mv_1^2/k_B$, and $\mathcal{M}_2 = 0.447$; the upstream kinetic energy is converted to internal energy in an adiabatic shock.

C. Oblique shocks

Now, consider *oblique shocks*; in which the upstream velocity $\mathbf{v}$ makes an angle $\beta < \pi/2$ with the shock surface. We know, from (9.7), that the tangential velocity component $u = v\cos\beta$ does not change in the shock. Thus, the normal upstream velocity, $w = v\sin\beta$, controls the jump conditions. It follows from this that an oblique shock exists only if

$$\mathcal{M}_1 \sin\beta \geq 1$$

This expression defines the *Mach angle*, μ_M: $\sin\mu_M = 1/\mathcal{M}$. (Check back to chapter 7 for another usage of the Mach angle). We expect, and will find, that w must decelerate in the shock; so $\mathbf{v}$ bends *towards* the shock face (away from the shock normal), due to the deceleration of the normal component of the velocity.

The jump conditions for an oblique shock can be found from the normal jump conditions, (9.12) & 9.13), by using $w_1 \sin\beta$ in place of w_1. Alternatively, looking ahead to MHD shocks, we can write the jump conditions explicitly in terms of components:

$$[\![\rho u]\!] = 0\,; \qquad [\![v]\!] = 0\,;$$
$$\left[\!\left[p + \rho w^2\right]\!\right] = 0 \tag{9.15}$$
$$\left[\!\left[h + \frac{1}{2}w^2\right]\!\right] = 0$$

It is also useful to define the deflection angle: $\delta = -[\![\beta]\!] = \beta - \beta_2$. This is the amount by which the flow bends *towards* the shock (or *away* from the shock normal). Geometry gives us, directly,

$$\frac{u_2}{u_1} = \frac{\tan\beta_2}{\tan\beta} = \frac{\tan(\beta-\delta)}{\tan\beta}$$

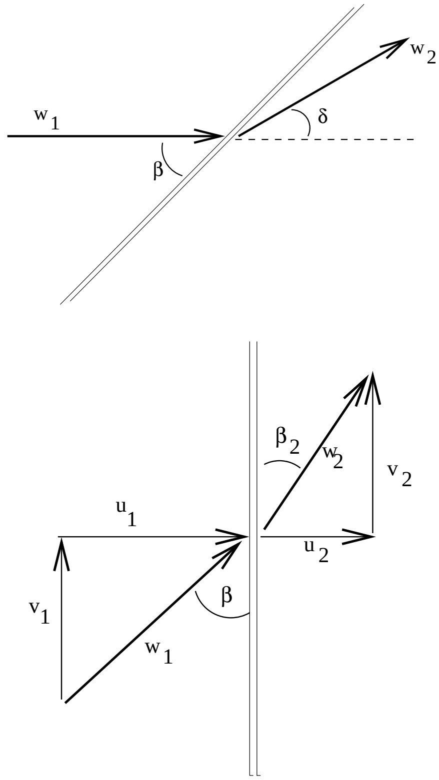

Figure 9.2. Oblique shock geometry. Top in terms of the incoming flow; $\mathbf{w}_1$ is the incoming velocity, at angle β to the shock; $\mathbf{w}_2$ is the outgoing velocity, and δ is the deflection angle, through which the flow bends at the shock. Bottom, the same thing with the shock rotated, and $\mathbf{w}$ divided up into components perpendicular and parallel to the shock face. This follows Faber, Figure 3.11, but note, our notation differs from his. In particular, we have subscripts 1 and 2 for upstream and downstream; he has unprimed and primed, respectively.

and the jump conditions give

$$-\frac{[\![w]\!]}{v} = \frac{\tan\delta}{\cos^2\beta\,(1+\tan\beta\tan\delta)} \tag{9.16}$$

We can use the jump conditions to find the usual post-shock quantities. If we start by specifying β, we can find relations analogous to (9.12 & 9.13)

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)\mathcal{M}^2\sin^2\beta}{(\gamma-1)\mathcal{M}^2\sin^2\beta+2} \\
\frac{p_2}{p_1} &= \frac{2\gamma\mathcal{M}^2\sin^2\beta}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \\
\frac{u_1}{u_2} &= \frac{(\gamma+1)\mathcal{M}^2\sin^2\beta}{(\gamma-1)\mathcal{M}^2\sin^2\beta+2}
\end{aligned} \tag{9.17}$$

(remembering that $\mathcal{M}$ refers to the upstream value, $\mathcal{M}_1$.) We can also find an expression for the post-shock mach number,

$$\mathcal{M}_2^2\sin^2(\beta-\delta) = \frac{(\gamma-1)\mathcal{M}^2\sin^2\beta+2}{2\gamma\mathcal{M}^2\sin^2\beta+(1-\gamma)} \tag{9.18}$$

Note that both $\mathcal{M}_2 > 1$ and $\mathcal{M}_2 < 1$ are possible.

1. TWO POSSIBLE DEFLECTIONS

We can also use the jump conditions to solve for the post-shock angles:

$$\tan\beta_2 = \tan\beta\frac{(\gamma-1)\mathcal{M}^2\sin^2\beta+2}{(\gamma+1)\mathcal{M}^2\sin^2\beta} \tag{9.19}$$

and

$$\tan\delta = 2\cot\beta\frac{\mathcal{M}^2\sin^2\beta-1}{\mathcal{M}^2(\gamma+\cos 2\beta)+2} \tag{9.20}$$

Figure 9.3 shows numerical solutions for $\delta(\beta,\mathcal{M})$, and Figure 9.4 shows a cartoon.

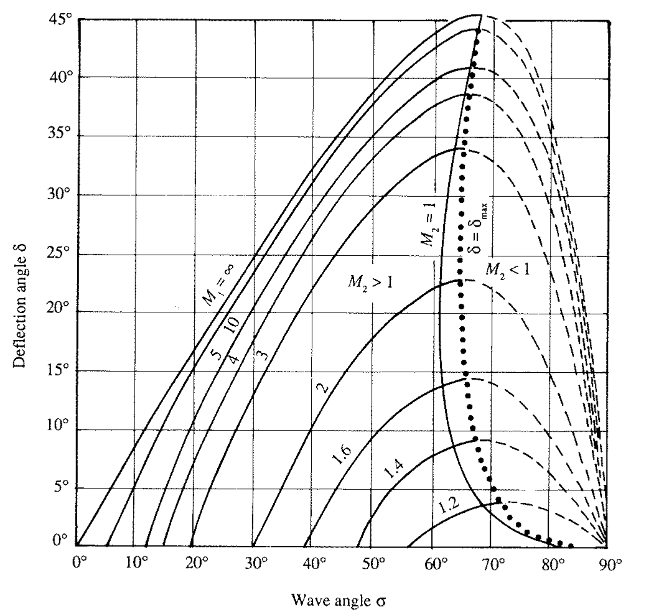

Figure 9.3. Oblique shock solutions. The dashed lines are the strong shock branch, the solid lines are the weak branch, and the heavy dotted line shows the maximum deflection angle δ_{max}. From Kundu figure 15.15. Note, σ in this figure is β in our notation.

This contains important results. First, the incoming angle, β, corresponding to a given deflection, δ, is double valued.[1] That is: for any specified deflection angle,

[1] This analysis gives us analytic expressions, (9.17, 9.19 and 9.20), for the post-shock conditions as functions of $\mathcal{M}$ and β. Alternatively, in shock tables, one may find $\mathcal{M}$ and δ specified; then there are two sets of post-shock conditions, corresponding to strong and weak shocks.

there are two possible shock angles (relative to the incoming flow). A given δ can come from (a) a **strong shock**, corresponding to high β values and giving subsonic postshock flow; and also a **weak shock**, from low β values and, generally, supersonic postshock flow. The flow will generally find the weak shock solution if it can.

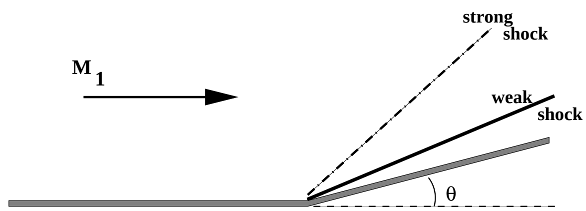

Figure 9.4. Weak and strong shocks. The geometry (θ) of the wall forces the flow to bend through θ; the mach number M_1 (and flow properties) determine whether forced bend will happen through a weak shock, a strong shock, or neither.

A second important result from this analysis is that there is a *maximum* deflection angle possible for any incoming flow. This connects to the the existence of detached shocks – that's what happens when the flow can't bend enough at an obstacle to form an attached shock. Figure 9.5 illustrates this.

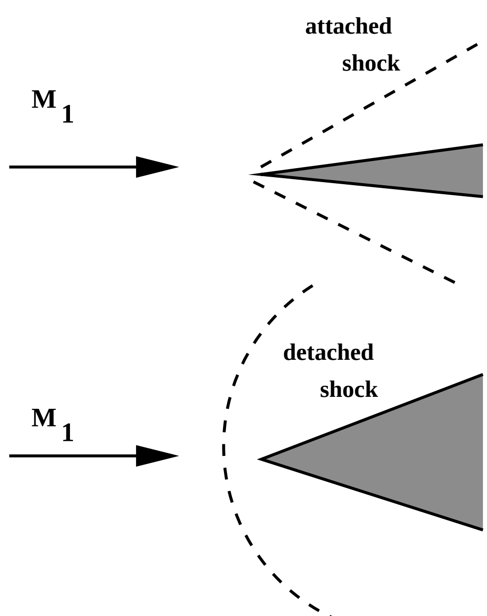

Figure 9.5. Attached *vs.* detached shocks. Top, the opening angle of the wedge is less than the maximum deflection angle of the flow; an attached shock forms. Bottom, the opening angle of the wedge is larger than the maximum possible deflection of the flow; a detached shock forms.

2. HIGH MACH NUMBER LIMIT

Finally, as one would expect, oblique shocks also have a $\mathcal{M} \gg 1$, "strong-shock" limit. The limits are just the same as for a normal shock, (9.14), if the normal velocity w is substituted for the full velocity v. The strong-shock limit of the deflection angle is still double-valued, and can be found from the high-$\mathcal{M}$ solutions in Figure 9.3.

D. The Weak Shock Limit

The weak shock limit describes shocks at oblique angles (close to the Mach angle) to the upstream flow, and thus shocks in which the velocity and pressure jumps across the shock are small. These "mach wave" shocks are important in bending supersonic flows.

For weak shocks, we can treat $[\![p]\!]/p_1$ as a small parameter. (Notation: Faber calls this σ, and does not necessarily assume it is small. I'm following Thompson here, in the weak-shock case.) To avoid notation confusion, here we let

$$\Pi = \frac{[\![p]\!]}{\gamma p_1}$$

(the γ factor simplifies the algebra later), and we assume $\Pi \ll 1$. With this, we can expand the jump conditions in Π, as

$$\begin{gathered} \frac{[\![\rho]\!]}{\rho_2} \simeq \Pi\,; \qquad \frac{[\![u]\!]}{c_{s1}} \simeq -\Pi \\ \frac{[\![c_s]\!]}{c_{s1}} \simeq \frac{\gamma-1}{2}\Pi \\ M_{1n} \simeq 1 + \frac{\gamma+1}{4}\Pi\,; \qquad M_{2n} \simeq 1 - \frac{\gamma+1}{4}\Pi \end{gathered} \tag{9.21}$$

Thus, both the incoming and downstream Mach numbers are close to unity. Seen in the lab frame, the shock is propogating at nearly the sound speed (as one would expect for a low-amplitude wave). We can also find the deflection and the shock angle, for weak oblique shocks

$$\begin{gathered} \delta \simeq \frac{\left(\mathcal{M}^2-1\right)^{1/2}}{\mathcal{M}^2}\Pi \\ \beta - \mu_M \simeq \left(1 - \frac{\gamma+1}{4}\right)\frac{\Pi}{(\mathcal{M}^2-1)} \end{gathered} \tag{9.22}$$

Thus, the flow deflection is small, and the shock angle (relative to the incoming flow) becomes the Mach angle, in the weak shock limit.

PRANDTL-MEYER FUNCTION

The relation between the deflection, δ, and the Mach number, $\mathcal{M}$, turns out to be a perfect differential (as we will show). The integral form of this function will be useful later.

Rearranging the weak-shock relations, we find to lowest order in $[\![u]\!]$,

$$\delta \simeq -\frac{\left(\mathcal{M}^2-1\right)^{1/2}}{\mathcal{M}^2}\frac{[\![u]\!]}{c_{s1}} \tag{9.23}$$

This is useful now. Since only the normal velocity changes at the shock, we have $w_1^2 - w_2^2 = u_1^2 - u_2^2$; and to first order in the jumps, $w_1 [\![w]\!] \simeq u_1 [\![u]\!]$. Thus, with $u_1/v \simeq \sin\mu_M = 1/\mathcal{M}$,

$$\frac{[\![w]\!]}{w_1} = \frac{1}{\mathcal{M}^2}\frac{[\![u]\!]}{c_{s1}}$$

Thus, we can rewrite (9.23) as

$$\frac{[\![w]\!]}{w_1} = \frac{\delta}{(\mathcal{M}^2-1)^{1/2}} \tag{9.24}$$

In the limit of infinitesimal strength, the oblique shock becomes a Mach wave, and with $\delta \rightarrow d\delta$, $[\![w]\!] \rightarrow dw$,

$$d\delta \simeq \pm\left(\mathcal{M}^2-1\right)^{1/2}\frac{dw}{w} \tag{9.25}$$

This can be written in terms of $\mathcal{M}$: using

$$\frac{d\mathcal{M}}{\mathcal{M}} = \frac{dw}{w} = \frac{dc_s}{c_s} \quad ; \quad wdw + \frac{c_s dc_s}{\Gamma - 1} = 0$$

where the factor $\Gamma = (\gamma+1)/2$ for a perfect gas. This gives

$$\frac{dw}{w} = \frac{d\mathcal{M}/\mathcal{M}}{1+(\Gamma-1)\mathcal{M}^2} \tag{9.26}$$

and, thus, we have expressed the deflection angle as athe perfect differential. This defines the *Prandtl-Meyer function*, $\delta(\mathcal{M})$:

$$d\delta = \frac{\left(\mathcal{M}^2-1\right)^{1/2}}{1+(\Gamma-1)\mathcal{M}^2}\frac{d\mathcal{M}}{\mathcal{M}} \tag{9.27}$$

This can be integrated formally. Taking the constant of integration so that $\delta(1) = 0$, the function describing the deflection angle, in the weak shock limit, as a function of total upstream mach number is

$$\delta(\mathcal{M}) = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left[\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\left(\mathcal{M}^2-1\right)^{1/2}\right] - \tan^{-1}\left(\mathcal{M}^2-1\right)^{1/2} \tag{9.28}$$

While we have only considered weak shocks, with $\mathcal{M} - 1 \ll 1$, we can formally look at the full range of this function. In particular, when $\mathcal{M} \rightarrow \infty$, the PM angle δ has a limit,

$$\delta_{max} = \frac{\pi}{2}\left[\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} - 1\right] \tag{9.29}$$

and for $\gamma = 5/3$, this happens to be just $\delta_{max} = \pi/2$.

References

I'm taking this mostly from Thompson, Kundu and Faber.