12 Accretion III: Disk models

We've already seen spherical (Bondi) accretion and talked about the general energetics of accretion flow. We now need a more concrete picture of disk accretion, which we expect to occur when the accreting gas has significant angular momentum.

The disk geometry may occur in at least two ways. The common picture is that of matter being dumped onto one member of a binary star system from its companion; the mass in this case clearly has significant orbital angular momentum and is confined more or less to a plane. Another possible case would be that of gas which is initially accreting spherically, with some angular momentum, and which can also lose energy radiatively. Radiative dissipation will reduce the internal energy and thus lead to a thin disk; but whatever dissipates the angular momentum of the gas may well be less efficient, so that the accreting gas remains supported by rotation in one plane, while cooling and flattening in the other direction.

Consider a binary star accretion disk, to be concrete. Mass will move from one star to the other when one of the stars (the companion) expands to fill its Roche lobe. The mass coming through the Lagrange point will have significant angular momentum, and will probably be initially in a non-circular orbit. However, it can lose energy quickly, by radiation, and will settle itself into a circular orbit (the lowest energy orbit for a given angular momentum). This will be a Keplerian orbit, with $v_\phi = (GM/r)^{1/2}$ (orbital velocity), and $l = (GMr)^{1/2}$ (specific angular momentum). This matter can only spread in radius if some of it loses angular momentum; this will happen, slowly, due to viscosity (as described below). Thus, if the mass flux stays fairly steady, the system will develop a steady accretion disk.

12.1 Models of thin (alpha) disks

One type of accretion disk model – the oldest, the first one developed – can be treated analytically. To start, we assume the gas in the disk moves in a circular orbit, at the local Kepler velocity, $v_\phi = r\Omega(r)$ where the angular speed is $\Omega(r) = (GM/r^3)^{1/2}$. The gas slowly drifts inward, at a radial/inflow velocity, $v_r \ll v_\phi$, and a mass accretion rate,

$$\dot{M} = 4\pi r^2 v_r \int \rho dz \qquad (12.1)$$

We will limit ourselves to physically thin disks; this means we can reduce the problem to one dimension. The vertical thickness of the disk is determined by hydrostatic equilibrium. This condition is, again, $\nabla p = \rho \mathbf{g}$, or $\partial p/\partial z = \rho g_z$. In the thin-disk limit, we write $\partial p/\partial z \simeq p/H$ if H is the disk scale height. For a disk which is dominated by the gravity of its central mass (M), rather than by its own self gravity, we have $g_z \simeq GMH/r^3$. Thus, the vertical support condition is

$$\frac{H}{r} \simeq \left(\frac{p}{\rho}\frac{r}{GM}\right)^{1/2} \simeq \frac{c_s}{v_\phi} \qquad (12.2)$$

so that a cool disk is a thin disk. The surface density is, then,

$$\Sigma = \int \rho dz \simeq \rho H \qquad (12.3)$$

Past this point, the analysis gets furry. There are many different models of accretion disks and accretion flows. The literature is a bit daunting, with a plethora of differing assumptions (and consequently differing results). However,

don't panic

The basic physics of steady accretion disks can be understood by looking at the simplest of the models that are out there, namely steady-state, spatially thin disks. These models are governed by a few simple principles: mass and momentum conservation and the effects of viscosity. However, even in the simplest model which these notes address, understanding how these basics affect the structure of the disk requires some algebra. I'm trying to lay out the argument in detail in these notes, following *Accretion Power in Astrophysics*, by Frank, King & Raine. Some of the important results in these notes are:

Mass conservation	equation (12.5), (12.6)
Viscous torques	equation (12.8), (12.9)
Angular momentum	equation (12.11), (12.12)
Inflow velocity	equation (12.21)
Alpha (α)	equation (12.22)
Accretion luminosity	equation (12.25)

So fasten your seatbelt ..

12.1.1 Mass conservation

Let's start simply. Consider a ring of disk material, lying between r and $r + dr$. It has total mass, $M(r, r+$

$dr) = 2\pi r\Sigma dr$. Now, this mass changes due to flows into and out of the ring; that is,

$$\frac{\partial}{\partial t}(2\pi r\Sigma dr) = v_r(r)2\pi r\Sigma(R) - v_r(r+dr)2\pi(r+dr)\Sigma(r+dr)$$

And thus, we have

$$\frac{\partial}{\partial t}(2\pi r\Sigma dr) \simeq -2\pi(dr)\frac{\partial}{\partial r}(rv_r\Sigma) \qquad (12.4)$$

Note we have assumed there are no local sources or sinks of mass, just the flows into and out of the ring. From this, as $dr \to 0$, we get the mass conservation equation:

$$r\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial r}(rv_r\Sigma) = 0 \qquad (12.5)$$

(Compare (4.2), our original form for mass conservation; can you see the connection to 12.5?). And: in a steady-state system, we have the expected expression for mass conservation:

$$\dot{M} = 2\pi r\Sigma v_r = \text{constant} \qquad (12.6)$$

Thus, as expected, $\dot{M}$ is constant with radius in a steady-state flow.

12.1.2 Viscosity and torque

What happens about the angular momentum? How does gas with finite angular momentum ever manage to move inwards? To answer this, consider a parcel of gas at r. It has specific angular momentum $l \propto r^{1/2}$, and this must be lost if the parcel is to move inwards. This is accomplished by the friction between rings of the differentially rotating disk. The friction is transmitted by viscosity.

> *Big fudge coming:* Formally viscosity is a microscopic process, transporting momentum "sideways" by particle collisions. We know how to treat this for a plasma, using Coulomb collisions as always. However, astrophysically particle-based viscosity is often very small (due to ionized plasmas being such good conductors), and turbulent viscosity will dominate. If the turbulence has mean velocity v_{turb} and mean scale λ, the coefficient of viscosity $\nu \sim \frac{1}{3}\lambda v_{turb}$. This is conceptually just fine, but very hard to write down analytically – so just about all accretion disk work makes a standard assumption (read "fudge"), as discussed below.

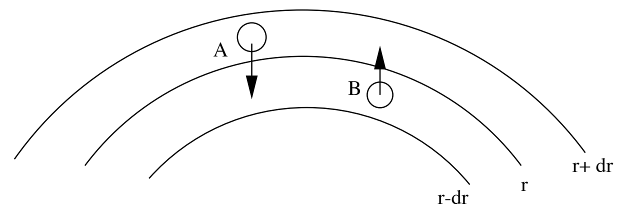

Figure 12.1 Viscous angular momentum transport in a shearing medium. Parcels A, B can be thought of either as single particles (for microscopic viscous transport) or as turbulent "eddies" (for turbulent viscosity). Following Figure 15 from Frank, King & Raine; connect notation to the text as $\lambda \leftrightarrow dr$

We now illustrate the effect of viscosity on angular momentum transport by considering a ring of matter at r, and two adjacent rings, at $r - \lambda$ and $r + \lambda$. The rings have $\Omega(r)$, $\Omega(r+\lambda) \simeq \Omega(r) + \frac{d\Omega}{dr}\lambda$, and $\Omega(r-\lambda) \simeq \Omega(r) - \frac{d\Omega}{dr}\lambda$. The scale λ can be any local differential, but it is most useful to connect it to the mean free path of whatever accounts for the viscosity.

To first order, viscous transport between the rings (which may be carried by single-particle collisions, or by more efficient turbulent motions) exchanges equal amounts of mass between the layers. We can write this mass flux as $2\pi\rho v_{turb} rH$ if v_{turb} is some characteristic turbulent or microscopic turbulent transport velocity. Note the rate of mass flux inwards must be the same as outwards, in a steady state disk. Now, consider an observer in corotation with the fluid on the surface $r =$ constant. The fluid at $r - \lambda/2$ will appear to move with velocity $v_\phi(r-\lambda/2) = (r-\lambda/2)\Omega(r-\lambda/2) - \Omega(r)r$. Thus the average angular momentum flux per unit lengh carried through r =constant in the outward direction is

$$\rho v_{turb} H\left(r - \frac{\lambda}{2}\right)\left[\left(r-\frac{\lambda}{2}\right)\Omega\left(r-\frac{\lambda}{2}\right) - r\Omega\right]$$

A similar expression, changing the sign of λ, gives the mean inward momentum flux per length. The difference between in and out gives the net outward momentum flux (which is also the torque per length). If we multiply this by $2\pi r$ we get the net torque exterted by the outer ring on the inner (and = - the torque of the inner on the outer, right?):

$$G(r) \simeq 2\pi r\lambda v_{turb}\Sigma r^2\frac{d\Omega}{dr} \qquad (12.7)$$

But now, we can collect λv_{turb} in the viscosity coefficient, ν, to write this as

$$G = 2\pi r^3\nu\Sigma\frac{d\Omega}{dr} \qquad (12.8)$$

where we've defined $\nu = \lambda v_{turb}$.

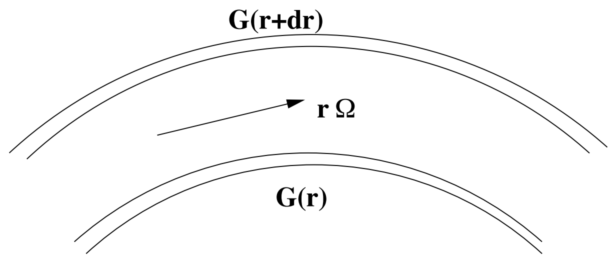

Figure 12.2 The net shear force on an annulus at r is the difference between the shear $G(r)$ on the outer and inner edges. This net shear force acts on the ϕ-component of angular momentum.

Finally, then, the net torque on a ring at r is the difference between the torque exerted by the inner and outer rings:

$$G(r+dr) - G(r) \simeq \frac{dG}{dr}dr \tag{12.9}$$

Thus, a Keplerian system in which $d\Omega/dr < 0$ has a net *outwards* flux of angular momentum. It is this which allows the inwards flow of matter, as a packet of gas slowly loses its l.

12.1.3 Angular momentum equation

Next: we need a conservation law for specific angular momentum. We'll use the same approach as we used for mass conservation – consider a ring of matter at r. It has total angular momentum $2\pi r(dr)\Sigma r^2\Omega$. This changes due to matter flowing in and out of the ring, and also due to the net torque on the ring (exerted because we've assumed each adjacent ring is in Keplerian motion, and because the matter is viscous). So we can repeat the same bookkeeping as above ... giving

$$\frac{\partial}{\partial t}\left(2\pi r(dr)\Sigma r^2\Omega\right) \simeq \\ - 2\pi(dr)\frac{\partial}{\partial r}\left(r\Sigma v_r r^2\Omega\right) + \frac{\partial G}{\partial r}(dr)$$

Taking the limit $dr \to 0$, this becomes exact, and we find the conservation law for angular momentum:

$$r\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{\partial}{\partial r}\left(r\Sigma v_r r^2\Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r} \tag{12.10}$$

Now – unlike the mass equation, the steady solution here leads to a "flow of angular momentum" which is a function of radius. In a steady state, (12.10) becomes

$$2\pi\frac{d}{dr}\left(r^3\Sigma v_r\Omega\right) = \frac{d\dot{J}}{dr} = \frac{dG}{dr} \tag{12.11}$$

where I've defined $\dot{J} = 2\pi r^2\Sigma v_r\Omega$ as the rate of angular momentum flow (outwards); note it is a function of r.

Now, integrate (12.11) over r:

$$\nu\Sigma\frac{d\Omega}{dr} = \Sigma v_r\Omega + \frac{C}{2\pi r^3} \tag{12.12}$$

which can also be written,

$$r^2\Sigma v_r\Omega = \frac{G}{2\pi} + C \tag{12.13}$$

Here, C is a constant of integration.

The conventional approach evaluates this at the inner boundary of the disk, at r_1 say. *If* we're talking about accretion onto a hard-surface star, r_1 is the stellar surface; the flow must approach solid-body rotation at the surface, so that $d\Omega/dr \to 0$ there. Alternatively, *if* we're talking about accretion onto a black hole, r_1 is taken as the minimum stable orbit. Inside of that the matter just "falls right across the event horizon", so that (it might be reasonable to assume) there is no torque on the last ring, $G \to 0$ there. In either case, this argument shows that the integration constant $C = -\dot{M}\left(GMr_1\right)^{1/2}$.

We get two important results from this. The first is a direct solution of (12.12):

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_1}{r}\right)^{1/2}\right] \tag{12.14}$$

This expression for the product $(\nu\Sigma)$ will be useful below. The second is an expression for the rate of angular momentum flow, $\dot{J}$. Going back to the definition, in (12.11), using (12.12) and the value of C, we get

$$\dot{J}(r) = -r^{1/2}(GM)^{1/2}\dot{M} \tag{12.15}$$

This thus shows explicitly that J flows *outward* when M flows *inward*, and also that $\dot{J}$ is a function of radius.

12.1.4 Accretion rate and radial velocity

Now that we have an expression for the viscous force, we can look at its effect on flow within the disk. We start with the basic equations of mass conservation,

$$r\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial r}\left(r\Sigma v_r\right) = 0 \tag{12.16}$$

and angular momentum conservation,

$$r\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{\partial}{\partial r}\left(r\Sigma v_r r^2\Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r} \tag{12.17}$$

Combining these, and noting $\partial\Omega/\partial t = 0$, we get

$$r\Sigma v_r \frac{\partial}{\partial r}\left(r^2\Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r} \quad (12.18)$$

(compare 12.11). We can also combine this with mass conservation to get

$$r\frac{\partial\Sigma}{\partial t} = -\frac{\partial}{\partial r}\left[\frac{1}{2\pi}\left[\frac{\partial(r^2\Omega)}{\partial r}\right]^{-1}\frac{\partial G}{\partial r}\right] \quad (12.19)$$

This result will give us the behavior of Σ, once Ω and G (which involves the viscosity ν) are specified. If we now assume Kepler orbits, (12.19) becomes

$$\frac{d\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(\nu\Sigma r^{1/2}\right)\right] \quad (12.20)$$

This last is the basic equation governing the time/space evolution of the density in a Keplerian disk. Solutions to this are not simple – they depend on the local viscosity (discussed below), and also on the local energetics (which we haven't even discussed – temperature and whatnot). When we do get a solution – and the literature is full of them – we can go back to (12.18) to find the radial/accretion velocity:

$$v_r = -\frac{3}{\Sigma r^{1/2}}\frac{\partial}{\partial r}\left(\nu\Sigma r^{1/2}\right) \quad (12.21)$$

Thus: the radial velocity (and thus $\dot{M}$, from equation 12.6) is determined by the local viscosity and the local surface density. We don't have enough information here to solve the system – that needs further arguments about internal energetics which I'm not going to go into. But we can see, directly from (12.21), that the inflow velocity $v_r \sim \nu/r$ (to order of magnitude).

12.1.5 What is ν?

The catch, of course, is that we do not know how to find the viscosity ν. This is the quantity which determines the rate at which matter can move inwards (the accretion rate); in addition, all of the details of the disk models (r-dependence of Σ, v_r, temperature, etc.) depend sensitively on ν. The common fudge in disk models is to parameterize the viscous stress in terms of the pressure, through a factor α = (viscous stress) / (pressure). Now, viscous forces come from adjacent rings of matter which are not moving at the same speed – so one slips against the other if $d\Omega/dr \neq 0$. Collecting all this, with the definition of α and the proper way to write the viscous stress, we get

$$\alpha p \simeq \nu\rho r\frac{d\Omega}{dr} \quad (12.22)$$

Remebering that ν has dimensions (turbulent velocity)×(turbulent length scale), or (mean particle speed)×(collision mean free path), you should see that (12.22) makes dimensional sense. This entire, grand hand-wave is collected in the term "α-disk". It's common to take $\alpha = 0.1$ in the literature.[1]

12.1.6 Energy dissipation and luminosity

One nice result is that we can find the radiated energy, and thus the efficiency ε, without knowing the details of α. To get this, we use the fact that shear stress dissipates energy. The local heating rate for this disk, integrated over the disk thickness – call it $D(r)$ (for dissipation) – is

$$D(r) = \frac{G}{4\pi r}\frac{d\Omega}{dr} = \nu\Sigma r^2\left(\frac{d\Omega}{dr}\right)^2 \quad (12.23)$$

(The first equality is the definition of heating by viscous shear stress; the second used the result 12.8 for G). From this, in a Kepler disk, we can find the local heating rate in terms of the basic quantities:

$$D(r) = \frac{3GM\dot{M}}{8\pi r^3}\left[1-\left(\frac{r_1}{r}\right)^{1/2}\right] \quad (12.24)$$

Now: where does this energy go? Viscous dissipation heats the local gas. *If the inflow speed is slow* – an important assumption in this type of disk model – then the gas must be able to radiate this away locally. That means we can equate $D(r)$ to the local luminosity per area from the disk. Integrating this over r, we find an expression for the luminosity:

$$L = \int_{r_1}^{\infty} D(r)2\pi r dr = \frac{GM\dot{M}}{2r_1} \quad (12.25)$$

Thus: if r_1 is small – as for a black hole ($r_1 \simeq r_{ms}$, the minimum stable orbit); or even for a neutron star – then the efficiency (refer back to 11.1) in this type of model can be high: $\varepsilon = r_s/2r_1$ where r_s is the Schwarzchild radius.

[1] Some order-of-magnitude estimates are perfectly valid in astrophysics. But perhaps one should not put overmuch confidence in the details of complicated, ornate accretion disk solutions – houses of cards – based on assumptions such as this α.

12.2 Extensions of the model

The basic, thin-disk, α-disk model in the previous section has been expanded and reworked extensively in the literature, in order to try to match the models to the data. Some of the extensions are as follows.

12.2.1 Hot and/or thick disks

In the previous model we assumed the disk is cold (that is, $c_s \ll v_\phi$), and therefore thin. This allowed a one-dimensional treatment. This isn't the only possibility, of course – the disk may be warm or hot, and therefore thick. Two versions of this exist in the literature. One is a "thick disk", a disk which is hot throughout. Such disks are still being worked on – their dynamics can be complicated, and they are probably globally unstable (they break up into non-axisymmetric structures – big "lumps" in the disk). Another variant is a cool, thin disk with a hot atmosphere ("corona"), or possibly a hot outflowing wind. This hot component does not radiate as a black body; it can be a source of high-frequency radiation (for instance γ rays, $h\nu \lesssim m_e c^2$).

12.2.2 Accretion flows

The α-disk model, above, was really developed for accretion onto a hard-surface star. Quite different inner boundary conditions, and inner flows, can occur if the disk sits around a black hole. The innermost gas can just slide across the event horizon. In such flows, the radial velocity, v_r, can go through a sonic point (much like Bondi accretion) before it reaches the last stable orbit. If this happens, a parcel of gas doesn't have time to radiate "locally" – so that the assumptions going into (12.23, 12.24) don't hold. These solutions – called ADAFs (for Advection Dominated Accretion Flows) – are therefore *underluminous*, with $L \ll GM\dot{M}/r$. Thus, an accreting black hole can be faint (relative to its Eddington luminosity) for at least two reasons. It may have a low accretion rate ($\dot{M} \ll \dot{M}_{edd}$); or its accretion flow may have a high $\dot{M}$ but be ADAF-like.

12.2.3 MHD effects

Everything we've done in this chapter has been field-free, *i.e.* purely hydrodynamic. That's probably unrealistic, because the accreting plasma is almost certainly magnetized. Think about the B field in the accreting material being "tied to infinity", to start. Flux freezing means that the field lines will be dragged along with the plasma; the field will be amplified, and will develop a toroidal component (forming sort of a helix). Two things follow from this. The helical field can channel, and even accelerate, a wind-type outflow ("MHD winds"). In addition, the field lines tied to the rotating plasma create a $\partial \mathbf{B}/\partial t$, which creates a $\mathbf{E}$ field: $\mathbf{E} = \mathbf{v}_{rot} \times \mathbf{B}/c$. This E field may be able to accelerate particles (are accretion disks a source of cosmic rays?). In addition, this system creates a $\mathbf{E} \times \mathbf{B}$ Poynting flux, also directed out along the rotation axis; this may be how jets are made.

Key points

- Thin accretion disk: the basic picture;
- Viscosity: what it is, why its important, how it controls $\dot{M}$;
- Energetics: what is the disk's luminosity?