2 Some basic plasma tools

In these notes I'm storing the ideas and important expressions for some basic plasma tools.

2.1 Distribution functions

This is an important tool for understanding the microphysics of a plasma: what is the *distribution function* (DF) of plasma particles (free charges) with momentum, or energy? In terms of momentum, this is defined so that $f(\mathbf{p})d\mathbf{p}$ is "the number of particles at $\mathbf{p}$; the total number of particles (usually per volume) is

$$n = \int f(\mathbf{p})d\mathbf{p} \tag{2.1}$$

If the particle distribution in momentum space is isotropic, the $d\mathbf{p}$ can be expanded out to give

$$\text{isotropic}: \quad n = \int f(p) 4\pi p^2 dp \tag{2.2}$$

You've probably seen the *Maxwell-Boltzmann* distribution, for a thermal, subrelativistic plasma:

$$f(p) = Ae^{-p^2/2mkT} \tag{2.3}$$

where the constant A is written in terms of the total number density n by plugging (2.3) into (2.2). We can also take suitable moments of the DF to get other useful things. For instance, the mean kinetic energy per particle, averaged over the MB distribution, is

$$\begin{aligned}\langle KE\rangle &= \frac{1}{n}\int A\frac{p^2}{2m}e^{-p^2/2mkT}4\pi p^2 dp \\ &= \frac{1}{2}mv_{th}^2 = \frac{3}{2}kT\end{aligned} \tag{2.4}$$

Note, the $1/n$ term is part of the definition of the mean energy (do you understand why?). This expression (2.4) also defines the *thermal speed*, $v_{th} = \sqrt{3kT/m}$ (this holds for monatomic particles; more complex molecules have more degrees of freedom & a different numerical factor).

We can also work with relativsitic plasmas.

• You recall that the total energy of a relativistic particle is given by

$$E^2 = p^2c^2 + m^2c^4 \tag{2.5}$$

we also have the definition

$$E = \gamma mc^2 \tag{2.6}$$

where

$$\beta = v/c \quad \text{and} \quad \gamma^2 = 1/(1-\beta^2) \tag{2.7}$$

In the limit $E \gg mc^2$, we also have $E \simeq pc$; thus the integrals in (2.1) or (2.2) can be written in terms of particle energy E (or just the Lorentz factor γ). An alternative DF, motivated by observations (for instance of cosmic rays) is often used with relativistic plasmas: the *power law* distribution,

$$f(E) = f_o E^{-s}\ , \quad E_1 \le E \le E_2 \tag{2.8}$$

or

$$n(\gamma) = n_o\gamma^{-s}\ , \quad \gamma_1 \le \gamma \le \gamma_2 \tag{2.9}$$

The exponent s depends on the system; the scaling constant f_o or n_o connects to the total number (or number density) of particles.

• NOTE the way these DF's are normalized: we want the total number of particles to be, say,

$$N = \int f(E)dE = \int n(\gamma)d\gamma \tag{2.10}$$

This means that $f(E)dE = n(\gamma)d\gamma$ .. so that $f(E)$ and $n(\gamma)$ have different units.[1]

2.2 Collective effects

Two important pieces of physics appear when we think about a "clump" of plasma.

2.2.1 Plasma waves

These will be treated more formally later on, but we can see the basics with a simple cartoon. Start with a layer of charge-neutral plasma, with number density $n_+ = n_- = n$. Now displace the electrons, relative to the (heavier) positive charges, by some distance ξ in the x-direction, as in the figure. The excess charge surface density is $ne\xi$; this generates an E field, $\mathbf{E} = 4\pi ne\xi\hat{\mathbf{x}}$.[2] Each charge layer feels a net force equal to its charge times the E field. Thus, we can write an equation of motion for the electrons,

$$m_e n\xi\frac{d^2\xi}{dt^2} = (ne\xi)(4\pi ne\xi) \tag{2.11}$$

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[1] To the student: what are those units?? Work out the dimensions – they will look funny, but that's the way it is.

[2] To the student: why?? think about Gauss's law and capacitors.

But this is clearly an equation for simple harmonic motion: $\xi(t) = \xi_o e^{-i\omega_p t}$, where ξ_o is the amplitude of the displacement, and

$$\omega_p^2 = 4\pi n e^2 / m_e \tag{2.12}$$

is the (square of the) *electron plasma frequency*. This is a fundamental mode of oscillation of the plasma; it's very easy to excite such waves, and in fact we expect any plasma to have some level, albeit low, of plasma wave turbulence.

We'll see later that ω_p is a cutoff frequency for EM wave propagation: only waves with $\omega > \omega_p$ can propagate in an unmagnetized plasma. We'll also see later that $\nu_p = \omega_p/2\pi$ is the frequency at which some plasmas emit coherent radiation (for instance in solar flares, or pulsar radio emission).

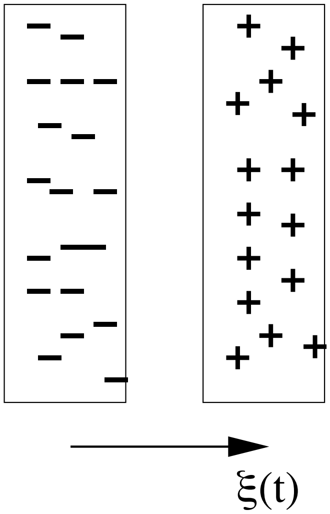

Figure 2.1 A simple cartoon illustrating plasma waves. The positive charges are displaced by $\xi(t)$ from the negatives; the attractive force turns this system into a simple harmonic oscillator with frequency ω_p.

2.2.2 Debye shielding

An important feature of plasmas is that their charges are very mobile; they can easily shield out any external E field we try to apply. Say we put a postive charge Q down somewhere in the plasma. Plasma particles of the opposite sign will scoot over to Q and form a charge cloud of the opposite sign around Q, thus neutralizing its effect on the rest of the system. If the plasma were cold (if thermal motions didn't matter), the charge cloud would be very thin around Q, and the shielding outside would be perfect. On the other hand, if the temperature is finite, particles on the edge of the cloud, where the E field is weak, have enough energy to escape from the potential well. Thus the nearby shielding is only partial – there is a region of finite size within which Q causes a finite E field. The size of this region is the *Debye length*; it is given by

$$\lambda_D^2 = kT/4\pi n e^2 \tag{2.13}$$

This is one of the important length scales in plasma physics.

One immediate use for λ_D is to turn it into $N_D = (4\pi/3)n\lambda_D^3$ – which measures the number of particles in a "Debye sphere". If $N_D \gg 1$, then Debye shielding is indeed a valid statistical concept, and we can treat the plasma as macroscopically charge neutral (that's what we'll do in this course). On the other hand, at high temperatures and/or low densities, it may be that $N_D \lesssim 1$, and we need to worry about single-particle motion as well as macroscopic effects.

2.3 Single particle motions

We also need to understand how individual particles move within the plasma.

2.3.1 Gyromotion

You have seen this before (right?). Just recall the basic analysis: the equation of motion for a particle with charge q, in a $\mathbf{B}$ field, is (in cgs!)

$$\frac{d\mathbf{p}}{dt} = q\frac{\mathbf{v}}{c} \times \mathbf{B} \tag{2.14}$$

For a subrelativistic particle, with $\mathbf{p} = m\mathbf{v}$, and putting the $\hat{\mathbf{z}}$ axis along $\mathbf{B}$, the solution to (2.14) describes gyromotion:

$$\begin{aligned} v_{x,y} &= v_\perp e^{i\Omega t} e^{i\phi} \\ v_z &= v_{zo} \end{aligned} \tag{2.15}$$

where v_{perp} is the (constant) amplitude of the motion across $\mathbf{B}$; v_{zo} is the (constant) velocity along $\mathbf{B}$; ϕ is the phase of the x or y motions; and the *gyrofrequency* is

$$\Omega = \frac{qB}{mc} \tag{2.16}$$

From this we can also get the *gyroradius* (also called Larmor radius),

$$r_L = \frac{mv_\perp c}{qB} \tag{2.17}$$

Or, in words: the general motion of a charged particle in a $\mathbf{B}$ field can be described as *gyromotion* about a *guiding center*, plus motion of the guiding center through space. In this simple case, the guiding center just moves along $\mathbf{B}$; but that will change in the next section.

- How does this change for a relativistic particle? In the homework, you will show that the gyrofrequency and gyroradius depend on the particle's energy, as

$$\Omega = \frac{qB}{\gamma mc}\,; \quad r_L = \frac{\gamma m v_\perp c}{qB} \tag{2.18}$$

2.3.2 Particle drifts, external forces

Now let's expand the example above, by adding an $\mathbf{E}$ field. The equation of motion is now

$$\frac{d\mathbf{p}}{dt} = q\frac{\mathbf{v}}{c} \times \mathbf{B} + q\mathbf{E}. \tag{2.19}$$

If $\mathbf{E}$ has components $(0, E_y, E_z)$ (Fig. 2.2) the solution for a subrelativistic particle is

$$\begin{aligned} v_x &= v_\perp e^{i\Omega t} - \frac{E_y}{B} \\ v_y &= i v_\perp e^{i\Omega t} \\ v_z &= v_{zo} + \frac{qE_z}{m} t \end{aligned} \tag{2.20}$$

Thus, we see (i) simple acceleration along $\mathbf{B}$, if $\mathbf{E}$ has a component in that direction; and (ii) sideways drift, *across* $\mathbf{B}$, at a rate

$$v_E = c\frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.21}$$

This is called "$\mathbf{E} \times \mathbf{B}$ drift". Comments:

- What causes the drift? Think about energetics: the particle alternately gains and loses energy, every half of its orbit. Thus, its Larmor radius gets alternately larger, then smaller. This leads to a net drift, as illustrated in the figure.

- We can also look at this particular drift in terms of Lorentz transforms. You may remember that $\mathbf{E}$ and $\mathbf{B}$ fields are not "relativistically pristine"; changing reference frames can turn $\mathbf{E}$ into $\mathbf{B}$, and vice versa. This connects directly to $\mathbf{E} \times \mathbf{B}$ drift. What direction do the particles go? Note, here, both positive and negative charges drift in the same direction. You can understand this from the cartoon; also note, $\mathbf{v}_E$ is independent of charge.

- We can generalize this. The key ingredient in what we just did, was the presence of a non-magnetic force (call it $\mathbf{F}$) in the equation of motion. Repeating the above analysis for this F, we find a generalized drift velocity,

$$v_F = c\frac{\mathbf{F} \times \mathbf{B}}{eB^2} \tag{2.22}$$

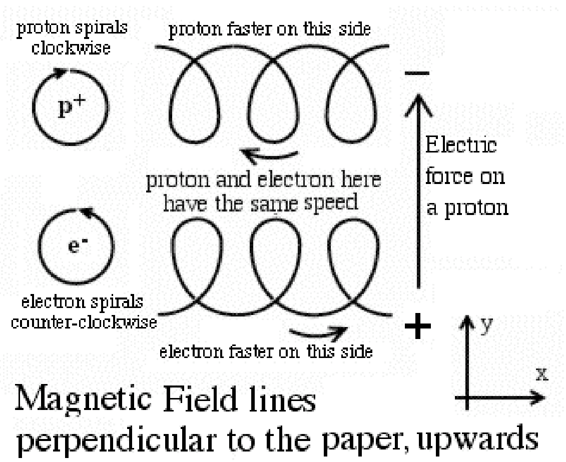

Figure 2.2 Illustrating $\mathbf{E} \times \mathbf{B}$ drift.

$\mathbf{F}$ can be anything relevant: gravity is the most common application.

2.3.3 Particle drifts, non-uniform B field

We also find drifts when a charge moves in a non-uniform $\mathbf{B}$ field. I'm not going to derive these formally (the algebra gets really tedious); rather we can do it by cartoon. One case is illustrated in Figure 2.3 – let $\mathbf{B}$ vary in space, and let $\nabla\mathbf{B}$ have a component perpendicular to $\mathbf{B}$. Once again, the Larmor radius of the particle will change during its gyro-orbit; and once again, this will cause a net drift across $\mathbf{B}$. One simple way to find the drift speed is to remember that a magnetic moment, $\boldsymbol{\mu}$, feels a force $\mathbf{F} = -\boldsymbol{\mu}\nabla\mathbf{B}$ when it sits in a nonuniform $\mathbf{B}$ field. You remember (of course....) that the gyromotion creates a magnetic moment (defined for a nonrelativistic particle),[3]

$$\mu = \frac{m v_\perp^2}{2B} \tag{2.23}$$

With this definition, the $\boldsymbol{\mu}\nabla\mathbf{B}$ force gives us the rate of "$\nabla\mathbf{B}$ drift":

$$\mathbf{v}_{\nabla\mathbf{B}} = \frac{v_\perp^2}{2\Omega}\frac{\mathbf{B} \times \nabla\mathbf{B}}{B^2} \tag{2.24}$$

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[3] why does this definition make sense? To derive this, you'll need to know that the definition of magnetic moment, in cgs, is $\mu = Ia/c$, for a current I going in a circle of area a.

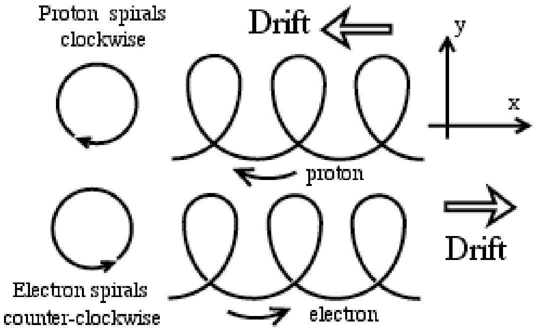

Figure 2.3 Illustrating ∇B drift. In this cartoon, $\mathbf{B}$ is out of the paper again; and it is stronger at the top of the drawing ($\nabla\mathbf{B} \parallel \hat{\mathbf{y}}$).

Another effect comes from a particle moving along a curved $\mathbf{B}$ line. Let $\mathbf{R}$ be radius of curvature of the field line, and define $\boldsymbol{\kappa} = \hat{\mathbf{R}}/R$.[4] The curved path creates a centrifugal force, and again causes a sideways *curvature drift*, at a rate

$$v_{curv} = -\frac{v_\parallel^2}{2\Omega}\frac{\boldsymbol{\kappa}\times\mathbf{B}}{B} = -\frac{v_\parallel^2}{\Omega}\frac{\mathbf{R}\times\mathbf{B}}{R^2 B} \qquad (2.25)$$

2.4 Adiabatic invariants

Let's continue with our particle in a nonuniform $\mathbf{B}$ field. It turns out that several useful constants of the motion can be found. We'll just look at one, the magnetic moment. Here's the result:

- If $\mathbf{B}$ is constant, or varies only slowly (compared to the gyroperiod), then μ is a constant of the motion.

Here's the outline of the proof. We're interested in nonuniform B fields. First, remember that if $\mathbf{B}$ changes with time (this is as seen by the particle in its gyro-orbit), it generates an EMF:

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{c}\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

so the rate of work done (current times EMF, right?) is

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{q\Omega}{2\pi}\frac{\pi r_L^2}{c}\frac{dB}{dt} \to \mu\frac{dB}{dt} \qquad (2.26)$$

But also, from (2.23), the definition of μ:

$$\frac{d}{dt}(\mu B) = \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) \qquad (2.27)$$

Now, compare (2.26) and (2.27): clearly these two are consistent only if $d\mu/dt = 0$; thus we've proved that μ is constant.

- I did this for a nonrelativistic particle. It can be generalized to the relativistic case: if we define

$$\mu_{rel} = \frac{\gamma m v_\perp^2}{2B} = \frac{p_\perp v_\perp}{2B} \qquad (2.28)$$

it can be shown that μ_{rel} is also a constant of the motion. So we can apply this analysis to cosmic rays – that's good.

2.5 Applications

Two applications are particularly interesing.

2.5.1 magnetic mirrors

This is a straightforward consequence of the invariance of μ. Think about a particle moving into a region of higher B (*i.e.*, converging field lines, in the usual cartoon). Because μ is constant, $v_\perp^2 \propto B$ must increase. But, the particle's energy is constant – so the gain in $v_\perp$ must come at the expense of $v_\parallel$. This clearly has a limit: when all of the particle's initial energy has been turned into $v_\perp^2$, there is no more $v_\parallel$, and the particle can't go any further along that field line. This point is called a *magnetic mirror*.

Mirrors are important for particle trapping .... if you want to keep a plasma confined magnetically, you might create it in a region of low B, which is bounded (at least along the field lines) by a region of high B. That's a "magnetic bottle". If each end of the confining field has a high-B region, particles are in principle trapped forever ... they can move back and forth along a field line (while undergoing gyromotion), but they can never escape the region (unless you add more physics ... as we'll talk about).

2.5.2 Particle acceleration

Magnetic mirror geometries can be used to accelerate the charged particles trapped therein.

For one method, think about a closed magnetic bottle, and now contrive to have the high-B regions approach each other. The trapped particles will gain a little bit of energy each time they collide with the mirror point (*e.g.*, think about a particle bouncing off a moving brick wall – go back to basic physics). This is called *Fermi acceleration*; it was the first mechanism proposed to accelerate cosmic rays.

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[4] Both $\mathbf{R}$ and $\boldsymbol{\kappa}$ are defined pointing *inward* relative to the arc of the circle (*i.e.* the curved field line).

For another method, let the magnetic bottle's geometry stay fixed (no moving end mirrors), but now let the B field go up and down with time, in some cyclic fashion. In the $dB/dt > 0$ phase, particles will gain perpendicular energy. If nothing else happens to the particles, their perpendicular energy will go up and down with the field (not very interesting). But what if the particles collide with each other before the field starts its downwards cycle? A collision between two particles will, statistically, redistribute energy between $v_\perp$ and $v_\parallel$. The parallel part of the velocity will not decrease when B goes back down – so the particle will have a net gain of energy on each cycle. This is *betatron acceleration* or *magnetic pumping*.

2.5.3 Earth's radiation belts

The region above the (mostly neutral) atmosphere, out to about 10 Earth radii, is more properly called the "inner magnetosphere" – but I'm using the older name, here, to differentiate from the full magnetosphere, which we'll visit later. The motion of particles in this region is mainly governed by single particle effects – gyromotion, $\mathbf{E} \times \mathbf{B}$ drifts, $\nabla \mathbf{B}$ drifts, and adiabatic invariants, such as we've seen in this chapter.

Some authors talk about three particle populations in this region. (1) The *cool, thermal plasma*, at particle energies below 100 eV, is mostly the magnetospheric extension of the ionosphere (*i.e.* terrestrial in origin). (2) The *ring current plasma*, particle energies from 100 eV up to several hundred keV, is "injected" into the radiation belts from magnetic storms in the Earth's magnetotail. Their drift motions result in a net current around the Earth, hence the name. (3) *Trapped radiation belt* or *van Allen belt* particles are even more energetic, at and above 1 MeV per particle. These are the ones whose radiation was detected in the 1950's (and thought to be a sign of nefarious, warlike activities on the part of other terrestrial nations); we now know these particles come from the solar wind, *via* the magnetotail.

Key points

- Relativistic particles: if you don't remember $p = \gamma\beta mc$, $E = \gamma mc^2$, etc., you should go back and review this material.
- Distribution functions: how they're defined; thermal *vs.* non-thermal.
- Plasma waves: what they are, what is ω_p?
- Debye shielding: what it is, what is λ_D?
- gyromotion: Ω, r_L, in cgs; for subrelativistic and relativstic.
- particle drifts: $\mathbf{E} \times \mathbf{B}$, $\nabla \mathbf{B}$, curvature drift.
- adiabatic invariants: μ is constant; magnetic mirrors.