3 Collisions in Plasmas

We want to understand the behavior of a collection of charges: that is, a plasma. We start with what a "collision" means for a set of charges. To begin, we recall the basics of hard-sphere collisions. If a "gas" of billiard balls, say, has a number density n and each particle has a random velocity v and a radius a, we define the collision cross section,

$$\sigma = \pi a^2 \tag{3.1}$$

From this we find the mean free path (the average distance between collisions),

$$\lambda \simeq \frac{1}{n\sigma} \tag{3.2}$$

and the mean time between collisions,

$$\tau_{coll} \simeq \frac{1}{n\sigma v} \; . \tag{3.3}$$

This last can be inverted to describe the collision rate per particle, $\tau_{coll}^{-1} \simeq n\sigma v$.

For hard spheres this analysis is straightforward, of course; they will not interact unless there is a direct "hit", and the geometrical cross section is the relevant one to describe energy exchange. Neutral atoms and molecules behave similarly, in that they need a very close hit; their cross sections can be calcalated from basic physics. Typical atomic cross sections $\sim 10^{-14}\mathrm{cm}^2$. Another "hard-sphere" type of calculation would describe direct hits between stars. Two stars must pass within a couple of stellar radii of each other for either of them to be strongly disturbed by the encounter; the cross section could be estimated from eq. (1), with $a \sim 2 - 3 \times R_*$.

3.1 The Spitzer collision cross section

There is, however, another type of encounter which is important in astrophysics: a long-range encounter between two objects which feel a $1/r^2$ force. This will describe collisions between charges in a plasma (an ionized gas), and will also describe distant collisions between stars (or any gravitating bodies). In addition it is the physical mechanism underlying bremsstrahlung radiation. I'm following the discussion in Longair, *High Energy Astrophysics, Vol. I*, chapter 2.

3.1.1 the basics

Start with a single encounter, in which particle A (an electron, say) scatters on particle B (a proton, say; with $m_p >> m_e$, we can assume the proton stays at rest. Let the incoming particle have velocity v and mass m_e, and let in come in at impact parameter b.

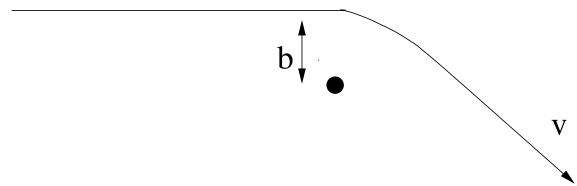

Figure 3.1 put a capation here

We can solve this problem exactly, from classical mechanics, and find the deflection angle, θ, and the resultant velocity and momentum changes, $\Delta\mathbf{p} = m\Delta\mathbf{v}$. Here, we will approximate this analysis, following Longair.

The net impulse on the electron will be $\Delta\mathbf{p} = \int \mathbf{F}(t)dt$, integrated over the collision. Now, the force is strong only when the two particles are close. Since they are close for a period of time $\Delta t \simeq 2b/v$, we can approximate $F \simeq e^2/b^2$ and $\Delta p \simeq 2Fb/v$. (Since we know the net deflection is perpendicular to the initial direction of motion, we can also drop the vector notation). This gives us the net energy gain per collision,

$$\Delta E = \frac{(\Delta p)^2}{2m_e} \simeq \frac{2e^4}{m_e b^2 v^2}$$

We want to extend this analysis, to find the net rate of energy exchange with the plasma. But the collision rate of our electron, with particles at impact parameter b is, (collisions/second) $= 2\pi nbv\, db$, we find the net energy exchange rate by integrating over all allowed b:

$$\frac{dE}{dt} = \int_{b_{min}}^{b_{max}} \frac{2e^4}{m_e b^2 v^2} 2\pi bnvdb = \frac{4\pi e^4 n}{m_e v} \ln\left(\frac{b_{max}}{b_{min}}\right) \tag{3.4}$$

Now, we want to express this in terms of a cross section:

$$\frac{dE}{dt} = \frac{E}{\tau_{coll}} = nv\sigma_c E \tag{3.5}$$

This defines the Coulomb cross section, σ_c:

$$\sigma_c = 8\pi \left(\frac{e^2}{m_e v^2}\right)^2 \ln\Lambda \tag{3.6}$$

if $\ln\Lambda = \ln(b_{max}/b_{min})$ is defined as the Coulomb logarithm.

The Coulomb logarithm depends on the largest and smallest impact parameters that are important (clearly, we cannnot integrate from $b_{min} = 0$ to $b_{max} = \infty$, since the integral in (3.4) would diverge). b_{min} is usually taken to be the distance corresponding to maximum energy transfer,

$$b_{min} \sim e^2/m_e v^2$$

b_{max} is less straightforward. Longair describes Coulomb scattering for an energetic particle hitting an electron bound in an atom, and for this case he likes $b_{max} \simeq v/\nu_o$, if $\nu_o = h^3/(2\pi)^2 m_e e^4$ is the electron's orbital frequency. (He argues that collisions slower than $1/\nu_o$ will violate the free-electron model assumed in the derivation). For unbound electrons, a more common choice is the Debye shielding length (the scale over which an extra charge causes charge separation in a plasma):

$$b_{max} \simeq \lambda_D = \left(k_B T/4\pi n e^2\right)^{1/2}$$

(k_B is the Boltzmann constant, and T is the temperature). Thus, the best choice of $\ln\Lambda$ clearly depends on the exact situation one is considering. Luckily, for our purposes, this is only a logarithmic uncertainty, and will not be critical for most of our calculations. The choices above, with typical astrophysical parameters, give $\ln\Lambda \simeq 10 - 20$, in almost any diffuse-matter setting.

Numerically, for a thermal plasma with $\frac{1}{2}m_e v^2 = k_B T$, the Coulomb cross section becomes,

$$\sigma_c \simeq 7 \times 10^{-13} \frac{\ln\Lambda}{T_4^2} \text{ cm}^2 \tag{3.7}$$

where $T_4 = T/10^4$K; so that

$$\tau_{coll} \simeq 4 \times 10^4 \frac{T_4^{3/2}}{n \ln\Lambda} \text{ sec} \tag{3.8}$$

and

$$\lambda \simeq 1 \times 10^{12} \frac{T_4^2}{n \ln\Lambda} \text{ cm} . \tag{3.9}$$

3.1.2 mnemonics and extensions

A useful short way to remember the Coulomb cross section is as follows. Similarly to the b_{min} estimate above, we can define an effective "radius", a_{eff}, by equating potential and kinetic energies:

$$\frac{e^2}{a_{eff}} = \frac{1}{2} m_e v^2 \tag{3.10}$$

and then, estimating $\sigma_c = 2\pi a_{eff}^2 \ln\Lambda$. This recovers the form of equaiton (3.6), and resembles the hard-sphere cross section, (3.1), "with a factor of $\ln\Lambda$ tacked on". The factor of 2 is retained in this estimate of σ_c, to match (3.6). In extending this to other examples, as we will do just below, the exact numerical factor that scales $\pi a_{eff}^2 \ln\Lambda$ cannot be recovered by this method of guessing; one would have to do a more formal analysis to get the correct order-unity numerical factor for each cross section.

Two other inverse-square-law cross sections can be immediately written down from this guesstimation. First, extend the Coulomb cross section to a relativistic plasma. If the particle energy is $\gamma m_e c^2$, where $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = v/c$, we get

$$\sigma_c \simeq 2\pi \left(\frac{e^2}{\gamma m_e c^2}\right)^2 \ln\Lambda \tag{3.11}$$

so that $\sigma_c \propto 1/E^2$ for this limit as well.

The other extension is to the cross section for energy exchange in a gravitating system (such as a star cluster). Here, we estimate a_{eff} from

$$\frac{G m_*^2}{a_{eff}} \simeq \frac{1}{2} m_* v^2$$

(we have assumed all of the stars have the same mass, m_*), and the gravitational cross section is

$$\sigma_{grav} \simeq 2\pi \left(\frac{2 G m_*}{v^2}\right)^2 \ln\Lambda \tag{3.12}$$

3.2 Anomalous effects

In the preceding section, we worked through a specific, well-formulated, very concrete example of particle collisions. That is, a free charge feels the electric field of an adjacent charge, or of all the nearby charges in the plasma, and changes its momentum and energy accordingly. This is attractive because we can write it down explictly. Unfortunately, it isn't always the whole answer. We know of several astrophysical situations where the predictions (for timescale, for instance) of Spitzer collisions disagree strongly with the observations.

• Plasmas in which the Spitzer collision time (or mean free path) is much longer than the characteristic age (or size) of the system are called *collisionless*. This is maybe an unfortunate terminology, because...

• ... "collisionless" doesn't really mean "no particle ever changes its energy or momentum due to collective effects". Rather, collisionless plasmas often "act collisional" (for instance they support shocks, which require dissipation). For a while this was not understood; now we believe that *plasma turbulence* is involved. That means that a random background of plasma waves (such as those discussed in §2.2.1) exists. A charged paraticle will scatter on the randomly fluctuating $\mathbf{E}$ fields associated with the turbulence; this will have the same effect as physical collisions, but (often) much faster than actual Spitzer/Coulomb collisions. Any such effect associated with plasma turbulence is called *anomalous* (resistivity, conductivity, *etc*. The collective effect of the turbulence is very hard to calculate from first principles; in these notes, we'll just assume that τ_{coll} refers either to Coulomb collisions, or anomalous effects, as needed.

3.3 Apply this: conductivity.

As an example of this, consider the conductivity in an ionized plasma. You recall that, in the simple case, the conductivity σ relates the current density to the local $\mathbf{E}$ field: $\mathbf{j} = \sigma \mathbf{E}$.

> *Notation alert.* Yes, I know, the Greek letter σ is doing double duty here, for the cross section and also for the conductivity. Sorry, but it's standard notation folks ...

3.3.1 isotropic conductivity

Let's start simply, ignoring the effects of any $\mathbf{B}$ field. We can then find σ simply, in terms of the mean time between collisions, τ_{coll} (or the collision frequency, $\nu_{coll} = 1/\tau_{coll}$), as follows. Consider a free electron, in a plasma, subjected to an external electric field E. The net force on the particle can be estimated,

$$F_{net} \simeq eE - \frac{\Delta p}{\Delta t} \tag{3.13}$$

where $\Delta p/\Delta t$ is the mean rate of momentum change per collision. But if the charges have a net drift velocity v_D, we can estimate $\Delta p/\Delta t \sim m_e v_D/\tau_{coll}$; then, in a steady state we have $F_{net} \simeq 0$, and the drift velocity must be $v_D = eE\tau_{coll}/m_e$. Next, we can use this in the (static) Ohm's law, to relate the conductivity to the drift velocity:

$$j = n_e e v_D = \sigma E \tag{3.14}$$

where the second equality defines σ. Collecting everything, we end up with

$$\sigma = \frac{n_e e^2}{m_e}\frac{1}{\nu_{coll}} = \frac{\omega_p^2}{4\pi}\tau_{coll} \tag{3.15}$$

(In the last expression, ω_p is the electron plasma frequency, which we've already seen). For our purposes here, the important fact is that $\sigma \propto \tau_{coll}$. Thus, in an ionized, low density plasma, collisions are infrequent, and the conductivity is very high.

3.3.2 anisotropic conductivity

Now, include a $\mathbf{B}$ field; in a general situation in which $\mathbf{E}$ and $\mathbf{B}$ exist. We expect single particle motion to have components along $\mathbf{B}$ (regulated by collisions, just as in the nonmagnetized case); across $\mathbf{B}$ (driven totally by collisions); and also $\mathbf{E} \times \mathbf{B}$ drift, in the third direction.

To start here, let's work out the single particle motion. We have

$$e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) - m\mathbf{v}\nu_{coll} = 0 \tag{3.16}$$

Put $\mathbf{B}$ along $\hat{\mathbf{z}}$, and write this in components:

$$\begin{aligned} eE_x + \frac{e}{c}v_y B - m\nu_{coll}v_x &= 0 \\ eE_y - + \frac{e}{c}v_x B - m\nu_{coll}v_y &= 0 \\ eE_z - m\nu_{coll}v_z &= 0 \end{aligned} \tag{3.17}$$

Now do some algebra, and solve for each component of $\mathbf{v}$:

$$\begin{aligned} v_x\left(1 + \frac{\Omega^2}{\nu_{coll}^2}\right) &= \frac{e}{m\nu_{coll}}E_x + \frac{\Omega^2}{\nu_{coll}^2}\frac{cE_y}{B} \\ v_y\left(1 + \frac{\Omega^2}{\nu_{coll}^2}\right) &= \frac{e}{m\nu_{coll}}Ey - \frac{\Omega^2}{\nu_{coll}^2}\frac{cE_x}{B} \\ v_z &= \frac{e}{m\nu_{coll}}E_z \end{aligned} \tag{3.18}$$

Or, this can be written in terms of the vectors $\mathbf{v}_\parallel, \mathbf{v}_\perp$, and $\mathbf{E}_\parallel, \mathbf{E}_\perp$ (relative to $\mathbf{B}$:

$$\begin{aligned} \mathbf{v}_\perp\left(1 + \Omega^2\tau_{coll}^2\right) &= \frac{e\tau_{coll}}{m}\mathbf{E}_\perp - \Omega^2\tau_{coll}^2 c\frac{\mathbf{E}\times\mathbf{B}}{B^2} \\ \mathbf{v}_\parallel &= \frac{e\tau_{coll}}{m}\mathbf{E}_\parallel \end{aligned} \tag{3.19}$$

Thus, single particle motion is a mix of direct flow along $\mathbf{E}_\parallel$, $\mathbf{E} \times \mathbf{B}$ drift, and a new effect, cross-field flow due to the collisions.

These three effects each contribute to a current. Remembering that in general, $\mathbf{j} = ne\mathbf{v}$ (for each charge species), the net current can be written,

$$\mathbf{j} = \sigma_o \mathbf{E}_{||} + \sigma_\perp \mathbf{E}_\perp + \sigma_H \hat{\mathbf{b}} \times \mathbf{E} \tag{3.20}$$

where $\hat{\mathbf{b}}$ is a unit vector along $\mathbf{B}$. We have *three* separate conductivities:

$$\begin{aligned} \sigma_o &= \frac{ne^2}{m\nu_{coll}} \,; \quad \sigma_\perp = \sigma_o \frac{\nu_{coll}^2}{\nu_{coll}^2 + \Omega^2} \\ \sigma_H &= \sigma_o \frac{\nu_{coll}\Omega}{\nu_{coll}^2 + \Omega^2} \end{aligned} \tag{3.21}$$

where $\Omega = eB/mc$ is the gyrofrequency, as usual. These three terms are called the collisional, Pederson and Hall conductivities. Comparing the two cross-field terms, we see that the Hall current dominates if Ω is large (so that gyromotion is much faster than collisions); or that the Pederson current wins if collisions dominate.

3.4 Apply this: diffusion

Another example is diffusion of one species into another (for instance, think about diffusion of cosmic rays into a thermal, subrelativistic part of the ISM).

3.4.1 isotropic diffusivity

We can pull the same trick as above, to estimate the diffusion rate for the isotropic case. Here, forget about any applied $\mathbf{E}$ field, but put a number of particles in a density gradient, ∇n. Keep the temperature constant, so that the density gradient connects to a pressure gradient, $\nabla p = kT\nabla n$. If the particles have number density n, and undergo collisions just as they did in the previous section, the net force per unit volume is

$$F_{net} \simeq -\nabla p - \frac{mnv_{diff}}{\tau_{coll}} \tag{3.22}$$

Note I've relabeled the drift velocity here, trying to clarify the notation.

> Why does equation (3.22) hold?? This can be proved in two ways. One way is to use macroscopic conservation laws – you know that pressure is a force per volume, so a pressure gradient exerts a net force on a unit volume of stuff. We'll do this formally in the next chapter. Alternatively, one can connect the micro to the macro by starting with a conservation law in phase space (called the Boltzmann equation) for all of the particles, multiply by $\mathbf{v}$ twice, and integrate over $d\mathbf{v}$ ... to get to the same point. We won't go through this second approach, but it does make clear the microscopic, statistical effect of a density gradient.

So, in this case the net force goes away for a diffusion-drift velocity, which we express in terms of the *flux* (particles/cm^2-s), as

$$nv_{diff} \simeq -\frac{kT}{m}\tau_{coll}\nabla n \tag{3.23}$$

We have, thus, a diffusion velocity that depends on the local density gradient – also note the sign (particles move towards lower density regions). This result is usually incorporated into the continuity equation:[1]

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\mathbf{v}) = \nabla \cdot (D\nabla n) \tag{3.24}$$

where

$$D = \frac{kT}{m}\tau_{coll} \tag{3.25}$$

is the *diffusion coefficient*. Note, D has dimensions of cm^2/s, and is often written $D \simeq v_{char}^2 \tau_{coll} \sim v_{char}\lambda$, where v_{char} is the "characteristic" speed (for instance thermal speed) of the particle distribution, and $\lambda = v_{char}\tau_{coll}$ is the mean free path of a particle. Thus, simple diffusion can be thought of as a random walk with step length λ.

3.4.2 anisotropic diffusivity

How does the presence of a $\mathbf{B}$ field change these results? About as you'd expect – particles have a much harder time diffusing across a field than along it. Refer back to the discussion around equations (3.16) through (3.18). We proceed similarly here, ignoring $\mathbf{E}$ but including a pressure gradient. Motion along $\mathbf{B}$ isn't changed; the equations of motion across $\mathbf{B}$ become

$$\begin{aligned} mnv_x\nu_{coll} &= -kT\frac{\partial n}{\partial x} + en\frac{v_y}{c}B \\ mnv_y\nu_{coll} &= -kT\frac{\partial n}{\partial y} - en\frac{v_x}{c}B \end{aligned} \tag{3.26}$$

[1] You've probably seen this before, most likely in E&M (remember conservation of charge). If not, trust me for a little bit, we'll derive in in the next chapter.

and after more algebra, we get to[2]

$$v_y\left(1+\frac{\Omega^2}{\nu_{coll}^2}\right)\simeq -\frac{D}{n}\frac{\partial n}{\partial y}$$
$$v_x\left(1+\frac{\Omega^2}{\nu_{coll}^2}\right)\simeq -\frac{D}{n}\frac{\partial n}{\partial x} \qquad (3.27)$$

(with the parallel motion unchanged). Similarly to the conductivity study, we thus have parallel and perpendicular diffusion coefficients:

$$D_{\|}=D=\frac{kT}{m}\tau_{coll}\ ;\quad D_{\perp}=\frac{D\nu_{coll}^2}{\nu_{coll}^2+\Omega^2} \qquad (3.28)$$

Thus – as with electrical conductivity – collisions slow down diffusion across $\mathbf{B}$. In the limit $\Omega >> \nu_{coll}$, the perpendicular diffusion coefficient becomes $D_{\perp}\simeq kT\nu_{coll}/m\Omega^2 = r_L^2\nu_{coll}$. Thus, cross-field diffusion can be thought of as a random walk with step length r_l (instead of λ).

Key points

• Collisions: mean free path, collision time, *etc*. – if you haven't used these before, make sure you understand them.

• "Spitzer" or "Coulomb" collisions: what they are, how to *estimate* the cross section.

• Anomalous effects: what they are, what we think they are due to.

• Electrical conductivity: how to build isotropic σ from basics; qualitative effects of $\mathbf{B}$.

• Diffusion coefficients: how to build isotropic D from basics; qualitative effects of $\mathbf{B}$.

[2]Doing the full analysis finds a third term, analogous to the $\mathbf{E}\times\mathbf{B}$ term in the anisotropic conductivity. It leads to a "density gradient drift", a.k.a. "diamagnetic drift", $\mathbf{v}_D \propto \nabla p\times\mathbf{B}$. It's usually slow compared to our diffusion, so I'm omitting it from these notes.