4 Basic fluid dynamics

Thus far in the course we've used a microscopic approach, emphasizing the effects of individual charges and single particle trajectories in the behavior of a plasma. Now, we move to a macroscopic approach. We want to describe the collective dynamical behavior of the system in terms of a few macroscopic variables – density, pressure, temperature, velocity, and (later on) currents and fields. This approach will be valid on scales large enough that we can ignore the discrete nature of the fluid (the fact that the gas, or plasma, or fluid, is composed of point-like particles). Thus, we must be working on scales large compared to the interparticle distance. In addition, some of our results will depend on collisions between the particles being important – this allows a shock to form, for instance. In these applications, we have the additional condition that the *effective collision length* must be small compared to the system size. This effective length can be (i) the Coulomb or hard-sphere mean free path; or (ii) the mean free path of a particle to "collisions" with microturbulence in the plasma (small-scale, small amplitude waves involving electric and/or magnetic field fluctuations); or (iii) the gyroradius of a charged particle, if the plasma contains a tangled magnetic field (in which case the particle motion across the field is severely restricted, and the effective "collision length" is somewhere between the gyroradius and the characteristic "tangling length" of the field).

Comments to the Reader.

In this chapter and the next I present the fundamental laws of astrophysical fluid mechanics: conservation of mass, momentum and (nearly) magnetic flux.[1] The fundamental relations are expressed as partial differential equations, and I have chosen (for the sake of having a thorough reference) to present them semi-formally. The down side of this is that they may look slightly intimidating to a student who does not often solve PDE's for fun and relaxation. I therefore offer a summary table – giving the most useful forms of the basic conservation laws, and some other critical results. These are the forms with which said student should be particularly familiar. There are also some simple, and important, applications coming. These important formal results can be found at:

Key (math) points

Mass conservation	eq. (4.2)
Momentum conservation	eq. (4.4)
Induction	eq. (5.11)

Key applications

Hydrostatic equilibrium	eq. (4.7)
Sound waves and sound speed	eq. (4.13)
Bernoulli's fact	eq. (4.18,19)
Flux freezing	eq. (5.14)
Alfven waves	eq. (5.9)

You might also note that the formal equations, while (of course!) necessary, are not the heart of the material at this level. Rather, the focus in class and in the homework will be on physical insight and simple applications of these basic laws.

4.1 Fluids: basics

Hydrodynamics starts with three basic equations, describing mass, energy and momentum conservation in the fluid.[2] We will consider the first two in this chapter. To extend to magnetohydrodynamics (MHD), we need a fourth basic equation, connecting the field to the fluid, and vice versa. That will come in chapter 5.

4.1.1 mass conservation

Consider an arbtrary volume of fluid, V, bounded by a closed surface, A; let the surface have an outward normal, $\hat{\mathbf{n}}$. The mass within this volume is $\int_V \rho dV$, if ρ is the mass density. The net rate of change of this mass is

$$\frac{d}{dt}\int_V \rho dV \quad ;$$

if there are no sources or sinks of matter, this quantity must equal zero. Now, there are two ways this integral can change with time. (i) there can be intrinsic variation of ρ, $\partial\rho/\partial t \neq 0$; or (ii) there can be flow into or

[1] We'll defer energy conservation to next term,after we've developed more tools.

[2] Terminology warning: hydrodynamics normally talks about "fluids"; but "gases" obey the same macroscopic laws, as do "plasmas".

out of the volume, at a rate $\rho\mathbf{v}\cdot\hat{\mathbf{n}}$ per surface area. The sum of (i) and (ii) must balance out to zero:

$$\frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial\rho}{\partial t}dV + \int_A \rho\mathbf{v}\cdot\hat{\mathbf{n}}dA = 0 \quad (4.1)$$

But the surface integral can be written as $\int_A \rho\mathbf{v}\cdot\hat{\mathbf{n}}dA = \int_V \nabla\cdot(\rho\mathbf{v})dV$. Since V is arbitrary, we can set the integrand to zero, and we get the differential form of this basic equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad (4.2)$$

This is, of course, the continuity equation, applied to mass conservation.

4.1.2 momentum conservation

Consider again our surface A, enclosing volume V. The momentum within this surface is $\int_V \rho\mathbf{v}dV$. The net rate of change of this quantity again must reflect intrinsic ($\partial/\partial t \neq 0$) variation and advection (flow across the surface). Thus, we write the net rate of change of momentum as

$$\begin{aligned}\int_V \frac{\partial}{\partial t}(\rho\mathbf{v})dV + \int_A (\rho\mathbf{v})\mathbf{v}\cdot\hat{\mathbf{n}}dA \\ = \int_V \frac{\partial}{\partial t}(\rho\mathbf{v})dV + \int_V \nabla\cdot(\rho\mathbf{v}\mathbf{v})dV\end{aligned} \quad (4.3)$$

In the second expression, we have used Gauss's law for tensors (noting that $\rho\mathbf{v}\mathbf{v}$ is a second-rank tensor).[3]

Now, the net rate of change of momentum in the volume must be equal to the net force exerted on the volume. We consider external forces which act throughout the volume ("body" forces, such as gravity, electomagnetism, bouyancy, radiation pressure if the fluid is optically thin; we let $\mathbf{f}$ be the net force per mass), and also the force exerted on the surface by the fluid outside V. The net force on the volume V is, then,

$$\int_V \rho\mathbf{f}dV - \int_A p\hat{\mathbf{n}}dA = \int_V \rho\mathbf{f}dV - \int_V \nabla p dV$$

where we have again used vector identities in the last step. If we take its differential form, expand the derivatives in the LHS and use (4.2) to simplify, we get

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \rho\mathbf{f} \quad (4.4)$$

This is our basic force equation (also known sometimes as Euler's equation, or the Navier-Stokes equation).[4]

4.1.3 lagrangian derivative

Look at the LHS of (4.2) or (4.4): both terms describe the "intrinsic" ways in which the mass, or momentum, in the elemental volume can change. It can be useful to collect them as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \quad (4.5)$$

which is called the "Lagrangian derivative". It describes the rate of change of whatever (mass, in 4.2; $\mathbf{v}$, in 4.4; etc) along the *trajectory* of the particle/fluid element. With this, the continuity equation becomes

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \quad (4.6)$$

and the momentum/Euler equation can be written similarly (simplifying to Cartesian geometry);

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \rho\mathbf{f} \quad (4.7)$$

4.2 Apply: hydrostatic equilibrium

After all that math, a couple of simple examples are in order. The first is a familiar one: consider Euler's equation, for force balance, in a fluid at rest (so that $\mathbf{v} = 0$). The most common application of this is in a gravitational field, $\mathbf{f} = \mathbf{g}$. This gives us just the condition for hydrostatic equilibrium:

$$\nabla p = \rho\mathbf{g} \quad (4.8)$$

[3] *Shriek!* you're probably saying... what the heck does the notation $\mathbf{ab}$ mean? In Cartesian coordinates, it's a 3x3 matrix, where the ijth component is constructed from the ith component of $\mathbf{a}$ and the jth component of $\mathbf{b}$:

$$[\mathbf{ab}]_{ij} = a_i b_j$$

I've added a page on "vector identities" to the class web page which has a bit more detail (in fact more than you'll need in this course).

[4] You should note that one important additional force term has not been included: the force between two adjacent fluid elements, due to the friction of their relative motion. This is the viscous term, and involves second derivatives of $\mathbf{v}$ in the space coordinates. We won't use this term in our examples, but it is often included in terrestrial applications.

4.2.1 planar atmosphere

For instance, if $\mathbf{g}$ is uniform, we can use the ideal gas law,

$$p = \rho \frac{k_B}{m} T \tag{4.9}$$

to write (4.8) as a DE in ρ. If $\mathbf{g} = -g\hat{\mathbf{z}}$, so that gravity is in the z direction, and if the gas is isothermal,[5] we can show this leads to the usual solution for the exponential atmosphere:

$$\rho(z) = \rho_o e^{-z/H} \tag{4.10}$$

where $H = k_B T/gm$. This is familiar as a description of the earth's atmosphere; it also applies to the ISM in the disk of the galaxy. Around our location in the plane, $\mathbf{g}$ is nearly vertical, so this describes the thickness of the ISM disk.

4.2.2 stellar equilibrium

Change the geometry to spherical: think about a star. The HSEq equation (4.8) becomes

$$\frac{dp}{dr} = -\rho \frac{GM(r)}{r^2} \tag{4.11}$$

where $M(r)$, the mass inside radius r, is of course

$$M(r) = 4\pi \int_0^r r^2 \rho(r) dr \tag{4.12}$$

So far so good. However, solving (4.11) is far from simple, because we can't assume the interior of the star is isothermal – so we have to introduce some further physics. That physics gets complicated: we must account for the energy sources within the star (from nuclear fusion), and how that energy is transported out from where it's generated (mostly by radiation, but also by convection). The details of the transport, combined with overall energy balance, determine the temperature structure inside the star ... folding that back into (4.11) eventually gives us the star's density structure.

Doing all that is very complicated, requires numerical solutions, and is far too much to go into in this course. However one simple scaling argument is useful. Look back at (4.11). We can approximate the LHS by

$$\frac{dp}{dr} \sim \frac{\Delta p}{\Delta r} \sim \frac{p_o}{R} \tag{4.13}$$

[5] BIG simplification here!!

if R is the star's radius and p_o is the gas pressure near the star's core. The RHS, evaluated at the star's surface, is M/R. Thus, remembering $\rho = nm$ (if m is the mean mass per particle), (4.11) can be approximated as

$$\frac{nkT}{R} \sim \frac{\rho GM}{R^2} \; ; \quad \frac{kT}{m} \sim \frac{GM}{R} \tag{4.14}$$

By this point, we're interpreting n, ρ, T as "typical" values in the core of the star. Cute result: the second expression is just energy balance. When the star is in hydrostatic equilibrium, its internal energy (per mass) is approximately equal to its potential energy (per mass).

That's a nice result; now let's consider what happens when HSEq is not satisfied.

4.2.3 star formation: gravitational instability

A star is a gravitationally bound system. Thus, the most fundamental idea is that a piece of the ISM can form a star when it is gravitationally unstable – as (Sir James) Jeans first pointed out. You have probably already seen an informal approach to this problem: if the (magnitude of the) gravitational potential energy exceeds the internal energy, the gas cloud (protostar) will collapse. For a cloud of radius R and fixed mass M, one way to express this is for a single particle of mass m:

$$\frac{GMm}{R} \gtrsim k_B T \; : \quad \text{or} \quad \frac{GM^2}{R} \gtrsim M \frac{k_B}{m} T \tag{4.15}$$

(if m is the mean mass per particle). This is clearly an *upper limit* on the size of a gravitationally unstable cloud. Alternatively, if we consider a piece of the ISM, we might want to hold the density fixed: the same criteria now becomes

$$\frac{4\pi}{3} G R^2 \rho \gtrsim \frac{k_B T}{m} \; ; \quad R \gtrsim R_J = \left(\frac{3}{4\pi} \frac{k_B T}{G m \rho} \right)^{1/2} \tag{4.16}$$

and

$$M \gtrsim M_J = \left(\frac{k_B T}{mG} \right)^{3/2} \left(\frac{3}{4\pi\rho} \right)^{1/2} \tag{4.17}$$

which is now a *lower limit* for the radius of an unstable region. This latter is usually identified with the original Jeans analysis: the length and mass scales in (4.16 and 4.17) are called the *Jeans' length* and *Jeans' mass*.[6]

[6] Comment from the author...please note where the apostrophe goes!

Free-fall time. What happens if the protostar is gravitationally unstable? How long does it take to collapse? If life is simple, this time is close to the *free-fall time*. Consider a piece of star at radius r, which suddenly loses gravitational support. Through "potential = kinetic" energy, its collapse speed will be $v_{ff}^2 \sim 4\pi G\rho R^2$; so the time it takes to fall a distance R is

$$t_{ff} \sim 1/\sqrt{G\rho} \tag{4.18}$$

You should also note that two important effects are not included in this simple approach: *rotation* and *magnetic fields*. Both will fight against gravitational collapse. We'll return to this in chapter 5.

4.3 Apply: Sound waves

This is an important concept: if you "hit" a fluid, how fast does the information (that the fluid has been hit) travel? Here's a physical approach, similar to the way we derived plasma waves.

Let some perturbation $(\delta\rho, \delta p, \delta T)$ be moving at some c_s. Ahead of the wave the fluid has $v = 0$; behind the wave the fluid has δv, in the same direction as the wave motion. Mass conservation at the wave front, in a frame moving with the wave front, gives

$$\rho c_s = (\rho + \delta\rho)(c_s - \delta v)$$

and to lowest order small, this gives

$$\delta v \simeq c_s \frac{\delta\rho}{\rho} \tag{4.19}$$

Thus, $\delta v > 0$ if $\delta\rho > 0$; the passage of a compression wave leaves behind a fluid moving in the direction of the wave. Now, apply momentum balance: the net force on some control volume, from the pressure difference, equals the rate of change of momentum in that volume. That is,

$$p - (p + \delta p) = \rho c_s \left[(c_s - \delta v) - c_s\right] \tag{4.20}$$

so that, again to first order small, we have

$$\delta p \simeq \rho c_s \delta v \tag{4.21}$$

Combining (4.19) and (4.21), we get a condition on the wave speed (to allow mass and momentum balancs):

$$c_s^2 = \frac{\delta p}{\delta\rho} \tag{4.22}$$

This is a big result: the fundamental signal speed in an unmagnetized fluid. Referring back to the ideal gas law, we also see that $c_s^2 \simeq kT/m$ (the "$\simeq$" describes the uncertainty in how T varies when ρ does).

4.4 Apply: the Bernoulli effect

You may have seen this before. It is basically an expression of energy conservation, for a moving fluid element. But you remember that one can derive energy conservation from momentum conservation.[7] Here, I go through the rather formal derivation for the hydrodynamic equivalent.

Start with Euler's equation, in the form (4.4). But now, note two useful facts. The first is that if the fluid is *barotropic* – that is if $p = p(\rho)$ only (as in an adiabatic gas), we have

$$\frac{1}{\rho}\nabla p = \nabla \int \frac{dp}{\rho} \tag{4.23}$$

(this can be verified using the chain rule; take $p = F(\rho)$, F being some function, and go from there). Thus, this term is a perfect differential. The second useful fact is that

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{v} \times \boldsymbol{\omega} + \nabla \left(\frac{1}{2}v^2\right) \tag{4.24}$$

(this is easiest to verify by expanding out in Cartesian coordinates). Thus, this term is also a perfect differential. The first term on the right hand side is written in terms of $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, the local *vorticity* (which is useful in advanced applications). Specify the force to gravity, which can be expressed in terms of a potential: $\mathbf{g} = \nabla\Phi_g$. If we then consider steady flow, we can rewrite (4.4) as

$$\nabla \left[\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi_g\right] = \mathbf{v} \times \boldsymbol{\omega} \tag{4.25}$$

But now: the right hand side of (4.25) is normal to both the local flow field (that is normal to streamlines) and to the local vorticity $\boldsymbol{\omega}$. Thus, we have one form of *Bernoulli's relation*: in inviscid, steady flow, the term in brackets has zero gradient in the direction of the local velocity field. We therefore have one version of Bernoulli's law:

$$\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi_g = \text{constant along streamline} \tag{4.26}$$

[7] Don't believe me? Start with "$F = ma$", let F come from the gradient of some potential, and integrate once; you'll get (kinetic energy) + (potential energy) = constant. Try it!

Further, in an adiabatic gas, $p \propto \rho^\gamma$ if γ is the adiatabic index (the ratio of specific heats). The second term simplifies, so that Bernoulli's relation for an inviscid adiabatic gas is

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \Phi_g = \text{constant along streamline} \tag{4.27}$$

Alternatively, in an incompressible fluid, ρ is constant, and the second term in (4.27) becomes simply p/ρ. Thus, for an incompressible fluid, Bernoulli's relation is

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \Phi_g = \text{constant along streamline} \tag{4.28}$$

4.4.1 example: free expansion

Now, we can apply this to the case of a piece of fluid expanding into a low-pressure environment. (For instance, this might describe a newly created HII region, which has suddenly been heated to $T \simeq 10^4$K, and has an internal pressure $>>$ that of its surroundings). If we are describing a cloud in the ISM, we can probably ignore gravity. First, we note that (4.27) can be written,

$$\frac{c_s^2}{\gamma - 1} + \frac{1}{2}v^2 = \text{constant} \tag{4.29}$$

Now, consider a spherical gas cloud, with finite density and pressure at its center, which is expanding into vacuum. This is clearly a time-dependent problem; but for times between its initial "violent" expansion, and its eventual final dispersion, we might get away with a nearly steady-state description. (That is, for these intermediate times, the rate at which the spatial dependence of the flow field changes is small compared to the rate at which an individual piece of fluid moves from the center to the outer edge). We can then relate conditions at the center of the cloud to conditions at the edge, by applying (4.29) at these two regions. Now, at the center, $v \simeq 0$ (since this is a spherical expansion, with a center at rest); and the cloud must have some finite central temperature, so that the central sound speed is c_{so}. Thus, the constant in (4.29) is $c_{so}^2/(\gamma - 1)$. Now, as $r \to \infty$, $p \to 0$, by assumption, so that $c_s \to 0$. (There can be conditions in which this last does not follow, if the external environment is very low-density but has a high internal energy per particle. However, such conditions probably do not hold in the ISM.) Thus, the constant from (4.29), evaluated at $r \to \infty$, must be $\frac{1}{2}v_\infty^2$. Finally, we can equate the constant evaluted at the center of the cloud, to the constant evaluated far away from the cloud, and infer

$$v_\infty^2 = \frac{2}{\gamma - 1}c_{so}^2 \tag{4.30}$$

Thus, a cloud having finite pressure, in a near-zero-pressure environment, will expand at its internal sound speed. (This makes sense; in the absence of driving forces, the best the cloud can do is to use its internal energy to drive the expansion; and c_{so} is a measure of this internal energy).

Key (physical) points

- mass conservation: how does the basic idea relate to the ways we use it mathematically?
- momentum conservation: ditto?
- hydrostatic equilibrium and gravitational (in)stability
- sound waves and the sound speed
- the Bernoulli effect, and how it applies to free expansion