5 Basic MHD

In this chapter we carry on with our approach of the previous chapter, but now include what's special to plasmas: their ability to carry currents, and to support and react to magnetic fields.

5.1 The Lorentz force

The effect of the $\mathbf{B}$ field on the force equation, (2.4), is straightforward. We simply add the Lorentz force to the momentum equation:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \frac{\mathbf{j}}{c} \times \mathbf{B} + \mathbf{F} \tag{5.1}$$

(where $\mathbf{F}$ is any other external force such as gravity, and I have ignored viscosity here). Now: expand out the Lorentz force as

$$\begin{aligned} \frac{\mathbf{j}}{c} \times \mathbf{B} &= \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= -\frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} \end{aligned} \tag{5.2}$$

This is an important breakdown of the Lorentz force; it demonstrates that the field exerts a **magnetic tension** and a **magnetic pressure** on the fluid. The first term in (5.2) represents the gradient of a scalar pressure, $p_B = B^2/8\pi$. It appears in the momentum equation parallel to the fluid pressure....you can think of trying to compress a magnetic field, by pushing at right angles to the field lines, with the field resisting the compression ("fighting back"). The second term is non-zero only if the field varies parallel to itself. A simple illustration is a curved field line. The curvature means there is a current flowing along the field line; the $\mathbf{j} \times \mathbf{B}$ force points inwards (relative to the curvature). Thus, curved field lines "want to straighten out"...Some authors combine both effects by describing magnetic field lines as "elastic bands within the fluid", which resist being stretched: either pushed together, or pulled transverse to their length.

5.2 Apply: plasma confinement

I don't know of any stars that are held together by magnetic fields[1] (think: can you come up with a spherically symmetric magnetic field that can confine a plasma? I bet not..) However, magnetic confinement is quite possible in other geometries. In fact, plasma confinement is the fundamental problem for laboratory plasmas, and may well be relevant to some astrophysical applications as well.

The issue is, can the plasma pressure can just balance the Lorentz forces from the fields? Most commonly, flows are ignored, as are resistivity and gravity. The general condition for equilibrium is, then,

$$\frac{\mathbf{j}}{c} \times \mathbf{B} = \nabla p \tag{5.3}$$

This is subject, of course, to the constraints

$$\nabla \cdot \mathbf{B} = 0\,; \qquad \nabla \cdot \mathbf{j} = 0\,; \qquad \mathbf{j} = \frac{c}{4\pi}\nabla \times \mathbf{B} \tag{5.4}$$

(The second relation holds in steady state, right?). A system which satisfies (5.3) also obeys

$$\mathbf{j} \cdot \nabla p = 0\,; \quad \mathbf{B} \cdot \nabla p = 0 \tag{5.5}$$

That is, constant-pressure surfaces are also "magnetic surfaces" and "current surfaces": $\mathbf{B}$ and $\mathbf{j}$ lines lie in constant-p surfaces.

Now... (5.3), with its auxilliaries (5.4) and (5.5), is "all" that is needed for laboratory confinement. We just have to solve it (and then test for stability). In these notes I confine myself to infinitely long plasmas in cylindrical geometry, which involve the simplest math, and may well be relevant to astrophysical jets. The basic equation, (5.3), becomes

$$\frac{dp}{dr} + \frac{d}{dr}\left(\frac{B_\phi^2 + B_z^2}{8\pi}\right) + \frac{B_\phi^2}{4\pi r} = 0 \tag{5.6}$$

Perhaps the most interesting application of this, for us, is the possibility that a current-carrying plasma can confine itself.

Example: linear pinch or z pinch Consider a purely azimuthal field: $\mathbf{B} = (0, B_\phi, 0)$, so that magnetic tension confines the plasma. The plasma can contribute to its own confinement by carrying just the right net current. The basic relation is

$$\frac{d}{dr}\left(p + \frac{B_\phi^2}{8\pi}\right) = -\frac{B_\phi^2}{4\pi r} \tag{5.7}$$

One possible equilibrium (illustrated in the figure) is $B_\phi(r) = B_o r/(1 + r^2/a^2)$ (exercise for the student: what are the corresponding pressure and current density profiles?) This type of pinch can be self-confining. Note that the current within radius r is

[1] although there does exist strong evidence that some molecular clouds, which are the sites of stellar birth in our galaxy, are held *up* by magnetic pressure ... that's for a later discussion.

$I(r) = \int_0^r 2\pi j_z r dr$, and from Maxwell the B field is $B_\phi(r) = 2I(r)/rc$. Using these and the pressure balance condition (5.7),

$$\int_0^a p r dr = \frac{1}{4\pi c^2} I_a^2 \tag{5.8}$$

where I_a is the current in the entire pinch (out to radius a), and we've assumed $p(a) = 0$. (To the student: can you derive this?). Thus, the plasma can self-confine if it carries the right current. This type of pinch is attractive, in that particles don't escape out the ends, and the current is carried by the plasma itself. However, this configuration turns out to be seriously unstable, thus is also of little practical interest in the lab.[2]

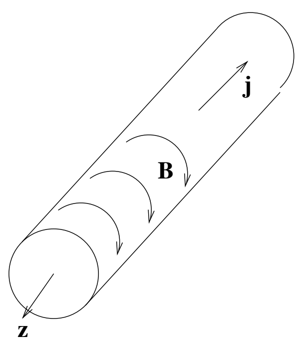

Figure 5.1a The geometry of a linear pinch: the field is azimuthal and the current is axial.

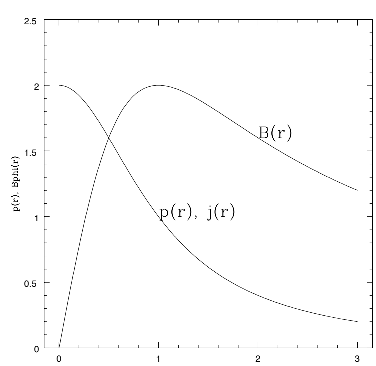

Figure 5.1b Qualitative solutions for magnetic field, pressure and current within a linear pinch.

Example: theta pinch We can switch the geometry above, and consider a "plasma solenoid". That is, run a current azimuthally around a cylinder of plasma – you will of course generate a B field parallel to the axis of the cylinder (as in Figure 5.2). This is not particularly interesting for astrophysics,[3] but it's common in laboratory plasmas (which live in tin cans). I'm including it here because I want to point to the figure later, when I discuss flux freezing.

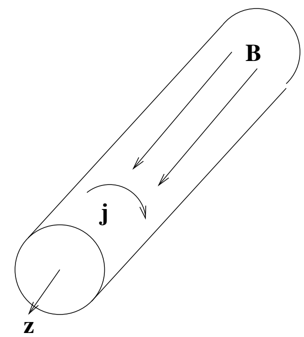

Figure 5.2 The geometry of a theta pinch: an azimuthal current supports an axial magnetic field.

5.3 Apply: Alfven waves

We saw above that any small disturbance in a non-magnetized gas will create sound waves. In a magnetized gas, one of the analogous waves (and probably the most important in astrophysical applications) is an Alfven wave. These waves also carry information, and (if the plasma is cold) are the signal-carrying waves.

What is an Alfven wave? These are waves in which the magnetic field dominates. It exerts the restoring force; fluctuations in the plasma density and pressure are either exactly zero, or unimportant. More specifically, an Alfven wave is a transverse wave, which is not compressive, and which propagates (in the simplest case) along the magnetic field. Thus, they can be thought of as propagating wiggles in the field lines, as in the figure.

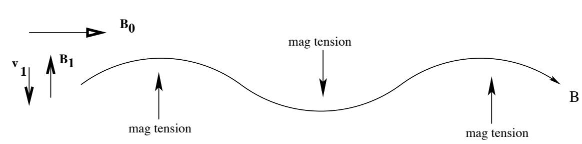

Figure 5.3 Schematic of simple Alfven waves; the perturbed B_1 and v_1 terms are perpendicular to the background field B_o. Following Cravens Figure 4.16.

Deriving the full details of the wave is long, but we can guesstimate the likely wave speed. Recall that waves in

[2]It might, however, be a useful model for radio jets – giving the plasma an axial velocity doesn't change the confinement physics. One must, however, think about how and where the circuit closes: how does the current return to the "origin" (the compact object that creates the radio jet)?

[3]Why? Can this geometry be self-confining, or does it need some external pressure to hold it together?

an elastic wire propagate due to the tension; as the field lines in a plasma exert a tension $B_o^2/4\pi$, one might expect a wave speed

$$v_A = \frac{B_o}{(4\pi\rho_o)^{1/2}} \tag{5.9}$$

This is the *Alfven speed*, and it is, indeed, a useful scaling speed for waves in a magnetized plasma. We can also note directly that $\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{B}_1 = 0$; so that the magnetic field perturbation must be normal to the wavevector.

5.4 The induction equation

Now we turn to the magnetic field in the fluid (which must be ionized, and thus a plasma, in order to interact with the field, right??)

Consider an arbitrary surface, S, within a fluid, bounded by some curve C. The magnetic flux within this surface is $\Phi_B = \int_S \mathbf{B} \cdot \hat{\mathbf{n}} dA$. We want to find an expression for $d\Phi_B/dt$. To get this, we start with Maxwell's equations; in particular the $\nabla\times\mathbf{B}$ and $\nabla\times\mathbf{E}$ ones. Also, we need Ohm's law for a moving fluid:

$$\mathbf{j} = \sigma \left[\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right] \tag{5.10}$$

where σ is the conductivity of the fluid or plasma. Now, if we take the curl of (5.10), and also note that

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2\mathbf{B} = \frac{4\pi}{c}\nabla \times \mathbf{j}$$

(since $\nabla \cdot \mathbf{B} = 0$), we find

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\nabla^2\mathbf{B} \tag{5.11}$$

where we've defined the *magnetic diffusivity*, $\eta = c^2/4\pi\sigma$. This describes the behavior of the magnetic field in a moving fluid with a specified conductivity. The first term describes induction due to the motion of the fluid, while the second (noting the second derivative) acts as a "diffusion" term, allowing field lines to "leak out"of high-field areas, for instance.

What happened to displacement current? Those of you who are fans of E&M will have noticed that there is no $\partial\mathbf{E}/\partial t$ term in the derivation of (5.11). This is a standard approximation in MHD; the reasoning goes as follows.

(1) Our fluids are very good conductors (why's that? Remember how ineffective Coulomb collisions are at dissipating currents), so we don't expect any free-charge $\mathbf{E}$ to stay around.

(2) Therefore the only $\mathbf{E}$ fields we expect are induced ones, which are $O(vB/c)$; thus $\partial\mathbf{E}/\partial t \sim O[(v/l)(vB/c)] \rightarrow O[(cB/l)(v^2/c^2)]$. But $v \ll c$ for sub-relativistic flows (almost always our limit here), so the displacement current is of order (small)^2, and we can ignore it.

5.4.1 Ideal limit: flux freezing

This is an important application; we'll use it a lot.

(a) Derivation. From the definition of Φ_B, we have

$$\frac{d\Phi_B}{dt} = \int_S \frac{\partial B}{\partial t} \cdot \hat{\mathbf{n}} dA + \oint_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) \tag{5.12}$$

where the second term, a line integral around the boundary of the surface, accounts for changes in the enclosed flux due to the motion of the surface. This line integral can be made a surface integral, and we find

$$\frac{d\Phi_B}{dt} = \int_S \left[\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B})\right] \cdot \hat{\mathbf{n}} dA \tag{5.13}$$

From this, and using (5.11), we find our desired result:

$$\frac{d\Phi_B}{dt} = \int_S \eta\nabla^2\mathbf{B} \cdot \hat{\mathbf{n}} dA \tag{5.14}$$

Thus, the rate of change of the magnetic flux depends on the inverse of the conductivity. In particular, astrophysical fluids are often highly conductive, so that $\sigma \rightarrow \infty$ and $\eta \rightarrow 0$. In that limit, we have $\Phi_B \simeq$ constant: the magnetic flux through some loop which is "tied to the plasma" is a constant.

(b) Do magnetic field fines exist? The concept of a magnetic line of force is an abstraction. In general no identity can be attached to these lines (they cannot be labelled in a varying field), nor can we speak of "motion" of field lines. In a perfect conductor, however, the concept of field lines becomes meaningful, due to flux freezing – and turns out to be a very useful way to envision what's going on.

Consider a material line in the fluid (say a chain of labelled droplets, or particles painted pink), defined by intersecting material surfaaces. Choose these surfaces everywhere tangential to $\mathbf{B}$ at $t = 0$. The flux through both surfaces is therefore zero to start, and their intersection defines a field line at that point. Flux freezing guarantees that these surfaces continue to satisfy $\Phi_B = 0$ at any later time. Thus, their intersection continues to define a field line, in fact the same field line – it has become identifiable; labelling the material (painting it pink) has labelled the field line, and the local fluid velocity $\mathbf{v}(\mathbf{x}, t)$ is also the velocity of that section of the field line. *The field line is attached to – "frozen into" – the fluid.*

(c) Flux freezing in practice. If we think of field lines as real entities, that move with the plasma, we can easily predict the effects of Lens' law on B fields in a moving plasma. Note, the easiest way to understand the physics, is to evaluate Φ_B over some imaginary surface *that is parallel to the field lines*.

For one example, think about a plasma cylinder with the B field along the axis, as in Figure 5.2. This might, for instance, be an astrophysical jet (never mind, right now, how the B field is maintained). The useful surface is just a cross-section through the cylinder. Now let the cylinder radius, R, increase – maybe the jet expands as it its source. Flux freezing means the product $B\pi R^2$ is constant; thus $B \propto 1/R^2$.

For another example, think about a plasma cylinder (or astrophysical jet) with azimuthal B – as in Figure 5.1a. The useful surface here might be a square, oriented with one edge along the jet axis, and the other edge along the outer surface of the jet. Now let the cylinder expand, R increase, but without any compression or stretching parallel to the axis. Exercise for the student: how does B vary with R now?

We'll see quite a few other examples as we go along – star formation, solar wind, the earth's magnetotail, neutron star formation in a supernova, accretion flows onto a black hole – all lean heavily on flux freezing.

5.4.2 resistive limit: flux annihilation

In a fluid with finite conductivity, flux freezing no longer holds. We can explore this by going to the other limiting case, when σ is small so that η becomes large. This is *diffusive limit*. If we simply ignore the advection term, equation (5.11) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} \tag{5.15}$$

This describes the effect of Ohmic dissipation on the magnetic field; note that it is a standard diffusion equation.

> **Do magnetic field lines diffuse?** We know how solutions to (5.15) behave: an initial field will decay on a timescale $\sim L^2/\eta$. Some authors discuss this in terms of field line "diffusion" or "slippage" out of the fluid. Remember that the density of field lines is related to the strength of the field; so a lower density of field lines, with time, should correspond to field lines "diffusing" out of the field. In particular, when η is finite, field lines are no longer tied to parcels of the plasma; some authors talk of field lines "moving through" the plasma in dissipative regions.

5.5 Protostellar collapse, revisited

OK, now let's return to our collapsing molecular cloud (MC). Assume that the MC somehow fragments into star-sized pieces. Next, consider the collapse of one such piece. The simplest physics will be when the initial cloud/protostar is nonrotating and unmagnetized. Let's pretend for now that this is the case.

A self-gravitating isothermal sphere is a useful model for this idealized collapse of the protostar. Because this is a simple HSEq solution, we might expect the collapsing cloud to want to find such a structure, as long as it can cool (and stay isothermal). The density structure of an isothermal sphere must obey

$$\frac{k_B T}{m} \frac{d\rho}{dr} = -\rho \nabla \Phi_g \tag{5.16}$$

where $M(r) = \int_0^r 4\pi r^2 \rho(r) dr$ and Φ_g is the gravitational potential (related to the gravitational field by $\mathbf{g} = -\nabla \Phi_g$). We must solve this in conjunction with Poisson's equation for gravity:

$$\nabla^2 \Phi_g = 4\pi G \rho \tag{5.17}$$

(Look familiar? Think about the analogous equation for the electric potential). When this is done, it turns out that one simple solution for the density is

$$\rho(r) \propto \frac{1}{r^2}$$

everywhere. Now, this clearly has two problems: $\rho(r) \rightarrow \infty$ as $r \rightarrow 0$, and $M(r) \rightarrow \infty$ as $r \rightarrow \infty$. Thus, real and finite clouds cannot satisfy this everywhere. This is not the only possible solution, however. A more satisfying physical solution has a core: the density is nearly constant (at ρ_o, say) for $r < r_o$, where $r_o^2 \propto T/\rho_o$. In addition, a satisfying physical solution must be truncated at large radii: $\int \rho(r) r^2 dr$ must converge.

Now, clouds collapsing under their own self-gravity can be modelled numerically. Such solutions of the collapse do find that the collapsing cloud moves through a series of nearly isothermal solutions – modified by a central density plateau, and by an outer edge (naturally; any simulation must be finite). Such a cloud will collapse "from the inside out" – since the higher central density will result in a shorter free-fall time (*cf.* 4.13). We would expect, then, that the core of the cloud would at some point become opaque to its own radiation, so that it can no longer cool; further collapse will heat the core. When the temperature reaches $\sim 10^7$K, nuclear burning will start and a star is born.

In the real world, however, we cannot neglect either the angular momentum or the magnetic field of the protostar.

Angular momentum If the collapsing cloud conserves its angular momentum, its angular velocity must increase as

$$\Omega(r) \propto \frac{1}{r^2}$$

Thus, a cloud which starts with only a slow rotation (for instance the differential galactic rotation,

$$\Delta\Omega_{cloud} \simeq \frac{d\Omega}{dR}\Delta R \sim \frac{\Omega(R)}{R} r_{cloud}$$

if R is the galactic radial coordinate and r_{cloud} is the cloud radius), will quickly speed up as it collapses. Without any loss of angular momentum, this spin-up will quickly provide centrifugal force which can balance the self-gravity: we would expect a large, rotating disk rather than a star. Clearly the initial angular momentum must be lost somehow in the collapse process. (Also, we note that the current angular momentum of the sun, for instance, $\ll \Delta\Omega_{cloud}(r_{cloud}/R_\odot)^2$; in agreement with this.)

Magnetic fields and flux freezing A simple picture also predicts that the contracting cloud will conserve magnetic flux. Referring back to (5.14), we recall that the magnetic flux, Φ_B, is nearly constant. This meanss the mean magnetic field in the cloud will increase, as

$$B \propto \frac{1}{r^2}$$

as the cloud colapses. As with angular momentum, the enhanced field will stop the collapse long before a star is reached; we can verify, again, that for the sun $B_\odot \ll B_{ISM}(r_{cloud}/R_\odot)^2$.

How are these problems resolved? We do not know the answer here, in detail. It is likely that a couple of processes are important. The first is the possibility of the conductivity being smaller than the discussion above suggests – which will happen if the cloud is mostly neutral. This will shorten t_{coll} and reduce σ – which will allow the flux inside the cloud to decrease, and the collapse to continue. One can picture the charged particles in the gas, to which the field is tied, "slipping past" the neutral gas as it collapses. This is called *ambipolar diffusion*. In order to estimate the time for the flux to change, t_{flux}, one must know the ion-neutral cross section; specific calculations suggest $t_{flux} \sim 10 - 100 \times t_{ff}$ for MC conditions.

This will reduce the magnetic flux within the cloud, and allow slow collapse. However, it does nothing for the angular momentum problem. The resolution of this is probably brought about by the torque exerted on the collapsing cloud by the magnetic field which threads the cloud. The magnetic field lines are very likely to connect to the ISM outside of the cloud (rather than to be contained wholly within the cloud); flux freezing in the ISM will tend to tie the "ends" of the field lines down, and they will thus exert a torque on the rotating, collapsing cloud. This will tend to slow the cloud down, and to transfer its angular momentum to the surroundings.

Key points

• Magnetic pressure and tension (when the field coexists with a plasma!)

• Self-confinement of a current-carrying plasma

• Flux freezing (when resistivity isn't important) and how it's applied.

• Magnetic "diffusion" or dissipation; when resistivity is important

• How do B fields affect protostellar collapse?