6 One-dimensional flows

In this chapter we'll continue exploring simple (well, fairly simple) examples of steady fluid/plasma flows.

6.1 The sound speed is important

In chapter 4 we introduced the sound speed: $c_s^2 = \partial p/\partial\rho$. There are important differences between subsonic and supersonic flows. Subsonic flows can be thought of as quasi-hydrostatic. That is, the flow field is strongly influenced by pressure gradients which are determined by conditions a long distance away (such as at boundaries). Supersonic flows, however, are quasi-ballistic. Pressure gradients have only a limited range of influence, and conditions far away have little or no effect on a solution locally.

The reason the sound speed is critical to the dynamics of a fluid or plasma flow, is that it is the speed at which information can propagate. We can illustrate this with 1D and 3D cartoons.

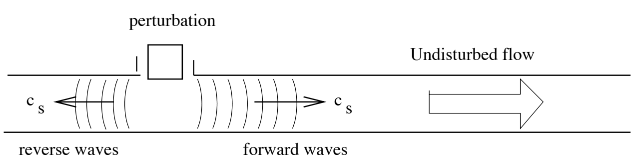

Figure 6.1 Physical illustration of simple waves. The information that the flow has been "whacked" at some point, propagates by simple sound waves, moving at speed c_s relative to the fluid in the pipe. Following Thompson figure 8.6.

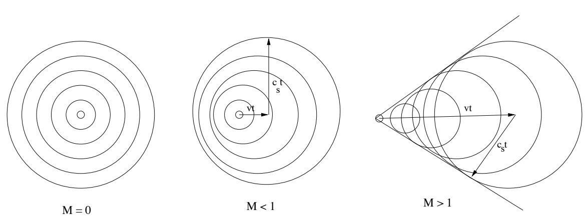

Figure 6.2 Mach's construction for the propagation of a disturbance. Consider a point source of sound (Thompson suggests a bumblebee) in a moving medium. If the source and flow are stationary, the sound propagates spherically from the source. If the source/flow are moving subsonically. the motion only distorts the spherical wavefronts. If, however, the motion is supersonic, all disturbances are confined to a *Mach cone*; an observer located outside of this cone does not receive any information about the bee.

We can also explore this by checking the magnitude of the terms in the (steady state) force equation:

$$\begin{array}{ccccc} \rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} & = & -\,\nabla p & + & \rho\mathbf{f} \\ (\rho v^2/L) & & (p/L) & & (\rho f) \\ (v^2/c_s^2) & & (1) & & (U_f/c_s^2) \end{array} \tag{6.1}$$

In the first line I've written the basic equation, in terms of some body force $\mathbf{f}$; in the second line I've estimated the magnitude of each term, for some scale length L; in the third line I've compared the three terms, using $p \sim c_s^2\rho$ and defining a "potential energy" or "work" associated with $\mathbf{f}$, $U_f \sim fL$. Thus: we see that the pressure gradient dominates the interial $(\mathbf{v}\cdot\nabla\mathbf{v})$ terms for subsonic flow, and vice versa for supersonic flow.

While subsonic flows can easily be smooth and continuous (with internal structure driven by a pressure gradient), it turns out that supersonic flows can (and usually do) contain discontinuous jumps in the flow properties (shocks).

6.2 Outflow: 1D channel flow

In general, a flow can't adjust smoothly from subsonic to supersonic; one or more shocks are generated by the transition. However, there are some special cases in which a smooth transition is possible.

Let's start with flow in a channel; and let the channel have cross section A. If A varies only slowly along the flow, then we can treat this is a 1D problem. If the flow is steady, mass conservation requires $\rho v A =$ constant (why?). Differentiating this, we find

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\frac{dA}{dx} = 0 \tag{6.2}$$

For the same flow, momentum conservation gives

$$\rho v\frac{dv}{dx} + \frac{dp}{dx} = 0 \tag{6.3}$$

(why? what terms have we retained or ignored in 6.2?). We need a third equation relating p and ρ: we use $c_s^2 = \partial p/\partial\rho$. Combining these results gives the basic equation for channel flow:

$$\left(\frac{v^2}{c_s^2} - 1\right)\frac{1}{v}\frac{dv}{dx} = \frac{1}{A}\frac{dA}{dx} \tag{6.4}$$

This is an interesting result. Consider the differences between subsonic ($v^2 < c_s^2$) and supersonic ($v^2 > c_s^2$) flow.

- Subsonic: a converging channel ($dA/dx < 0$) accelerates the flow (leads to $dv/dx > 0$), and a diverging channel ($dA/dx > 0$ decelerates it ($\Rightarrow dv/dx < 0$).

- Supersonic: a converging channel ($dA/dx < 0$) *decelerates* the flow (leads to $dv/dx < 0$), and a diverging channel ($dA/dx > 0$ *accelerates* it ($\Rightarrow dv/dx > 0$).

Consider, then, a flow which starts subsonic in a converging channel. It will accelerate as the channel narrows. If things are set up *just right*, the flow will reach $v = c_s$ just at the narrowest point of the channel. If the channel broadens again, the flow can accelerate smoothly to supersonic speeds. Referring to (6.3), we see that "just right" means the flow must reach $v = c_s$ exactly at the narrowest point of the channel. If this is not the case, the flow can do one of several things. It can (i) change from acceleration to deceleration (or vice versa); (ii) it may not be able to remain steady; or (iii) it may set up internal shocks to enable it to adjust to the local conditions in the channel.

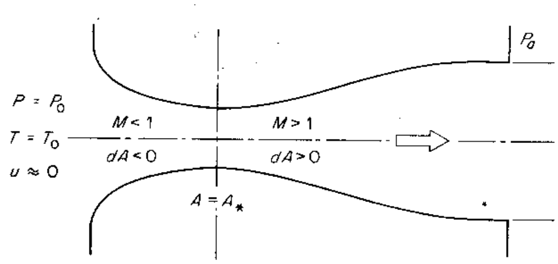

Figure 6.3 Transonic flow in a convergent-divergent nozzle. If the throat occurs at just the right place, relative to the flow, then a smooth transition from subsonic to supersonic is possible. If, however, the flow is not exactly at $v = c^2$ when it reaches the throat, it cannot remain steady: shocks and/or time-unsteady flow happen. From Thompson figure 6.3.

6.3 Outflow: stellar winds

Spherical stellar winds – taking the solar wind as a well-studied example – are an interesting extension of channel flow. We will break the problem into two parts: first, demonstrate that the extended atmosphere of the sun can't be static; and second, the nature of the solar wind outflow. We'll return to the interaction of the flow with a cooler, finite-density ISM at a later point.

6.3.1 Why must there be a solar wind?

We know, from simple observations, that the sun has a hot atmosphere (the chromosphere, and the more extended corona). Are static solutions possible for the solar atmosphere? These would be solutions of

$$\frac{dp}{dr} = -\rho \frac{GM_\odot}{r^2} \tag{6.5}$$

If the temperature structure of the atmosphere $T(r)$, is known, then (6.4) can be integrated easily. We can consider two simple cases:

• First, think about an isothermal atmosphere. In this case, solutions of (6.5) predict a *finite* pressure at infinity:

$$p_\infty = p_\odot \exp\left(-\frac{GM_\odot m_p}{k_B T_\odot R_\odot}\right) \tag{6.6}$$

where $T_\odot \simeq 1.5 \times 10^6$K and $p_\odot \simeq 0.3$N/m^2 are the temperature and pressure at the base of the corona. Evaluating this limit, one finds that $p_\infty \gg p_{ISM} \simeq 10^{-8}$N/m^2. Thus: a static atmosphere would have a pressure at infinity that greatly exceeds the surrounding pressure of the ISM – and so it can't be static. The solution we've just derived would have a tendency to expand outward.

• Can we devise another $T(r)$ profile to alleviate this? Not easily ... we might expect if the temperature falls rapidly enough outwards, one might come up with an acceptible p_∞. However, the temperature gradient can't be steeper than that allowed by thermal conduction (heat flow from the hot solar surface). This turns out to be $T(r) \propto 1/r^{2/7}$; which still has an overly large p_∞.

Thus, we need to look at a *dynamic* solution of the momentum equation – one which gives an outflow or "wind".

6.3.2 The basic wind solution

Next, how does the wind behave (ignoring magnetic fields for the time being)? The basic solution is due to Parker. Consider a steady, spherical outflow. Mass conservation in this case is $\rho v r^2$ =constant; or,

$$\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{v}\frac{dv}{dr} + \frac{2}{r} = 0 \tag{6.7}$$

while the momentum equation becomes in this case (noting that gravity from the central star is important),

$$\rho v \frac{dv}{dr} + \frac{dp}{dr} = -\rho \frac{GM}{r^2} \tag{6.8}$$

Writing $dp/dr = c_s^2 d\rho/dr$, these two equations combine to give the basic wind equation,

$$\left(v - \frac{c_s^2}{v}\right)\frac{dv}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2} \tag{6.9}$$

This does not have analytic solutions over the whole range of r. However, we can learn quite a bit about the nature of the solutions simply by inspection of (6.9), as follows.

• The left hand side contains a zero, at $v^2 = c_s^2$. If we want to consider well-behaved flows, that is to say those in which the derivative dv/dr does not blow up, then the right hand side of (6.9) must go to zero at the same point. This defines the condition that must be met at the sonic point:

$$v^2 = c_s^2 \qquad \text{at} \qquad r = r_s = \frac{GM}{2c_s^2} \tag{6.10}$$

Whether or not a particular flow satisfies this condition depends on the starting conditions, such as with what velocity and temperature it left the stellar surface, and also what the boundary conditions at large distances are. If it does not start in such a way to satisfy this condition, it either stays subsonic (corresponding to finite pressure at infinity), or cannot establish a steady flow.

• The solution beyond the sonic point depends on the temperature structure of the wind. The only solutions with $dv/dr > 0$ for $r > r_s$ are those for which $c_s^2(r)$ drops off more slowly than $1/r$; it is only these for which the right-hand side stays positive. In the case of an isothermal wind, with c_s^2 = constant, (6.9) can be solved in the limit $r \gg r_s$:

$$v^2(r) \simeq 4c_s^2 \ln r + \text{constant} \tag{6.11}$$

Thus, the wind will be supersonic, by a factor of a few, as $r \to \infty$. The question of how the solar wind manages to stay nearly isothermal is not solved; it is probably due to energy transport by some sort of waves (MHD or plasma waves, for instance) which are generated in the photosphere and damped somewhere far out in the wind.

• Inside the sonic point, the gravity term will dominate the right hand side of (6.9). Thus, solutions with $dv/dr > 0$, and $v^2 < c_s^2$, will obey

$$\frac{c_s^2}{v}\frac{dv}{dr} \simeq -\frac{GM}{r^2}$$

This equation looks as if gravity is driving the wind out! This unlikely-looking result comes from the fact that the flow is nearly subsonic in this region; therefore, the dp/dr term in (6.9) – which actually drives the wind out – is nearly equal to the gravity term.

6.3.3 What about MHD effects?

How can we justify ignoring the magnetic field in this analysis? To explore this, go back to the basic momentum equation (4.4), write it for steady flow, and estimate the magnitude of each term:

$$\begin{array}{ccccccc} \rho\mathbf{v}\cdot\nabla\mathbf{v} = & -\nabla p & + & \dfrac{\mathbf{j}}{c}\times\mathbf{B} & - & \rho\dfrac{GM_\odot}{r^2} \\ (\rho v^2/L) & (p/L) & & (p_B/L) & & (\rho GM/r^2) \\ (v^2/v_A^2) & (c_s^2/v_A^2) & & (1) & & (U_{grav}/v_A^2) \end{array} \tag{6.12}$$

In the first line of (6.12) I've written the real equation. In the second line I've estimated the magnitude of each term (for some scale length L, and gravitational potential energy U_{grav}), and in the third line I've compared the relative magnitudes of each term to the Lorentz force term. Thus: we can ignore MHD effects when the Alfven speed is low – when the B field is low, or when the density is high.

Looking at numbers we know for the solar wind, this limit holds (i) very close to the sun, where the density is high; and (past about 10 solar radii, where $v \sim v_A$ and $U_{grav} \ll v_A^2$. Inbetween these limits, in the range $\sim 2 - 10R_\odot$, we find that the $\mathbf{j}\times\mathbf{B}$ term dominates. It is in this region that the B field controls the geometry of the flow. This is the region in which the field is changing between "emerging flux ropes", fully connected to the sun's surface, and open field lines, connected to the solar wind; the plasma is constrained to flow along open field lines, and thus the configuration of these open field lines determines the "area function" that channels the plasma flow. Finally, past $\sim 10R_\odot$, the flow is strongly superalfvenic as well as supersonic; the plasma is capable of "pushing the field around". We can use flux freezing here, and think of the field lines as being stretched out by the outflowing, supersonic plasma.

The outer, intertia-dominated regions of the solar wind include close to earth, where space probes provide us with good, detailed information about the nearby structure of the wind. We know that the equatorial plane of the solar wind contains four *sectors*, two with outwards-pointing magnetic field, and two with inwards-pointing field. We also know that the global field direction reverses direction above and below the equatorial plane (as you would expect when the basically dipolar magnetic field of the sun is stretched out by the solar wind). Thus, the equatorial plane must contain a *current sheet* (think: which way must the current flow?). It turns out that this current sheet is also the explanation of the sectors.

6.3.4 What about shocks?

Just a quick note here. The solar wind flow is highly supersonic, and superalfvenic, by the time it reaches earth. But here and there it must slow down: when it is forced to go around a planet, and when it runs into the local ISM. We expect a shock transition at each of these sites. We find a *bow shock* where the solar wind encounters the earth's magnetosphere – and similar structures around the other planets. We also find an *outer shock* – called the *heliopause* – where the solar wind's pressure has dropped to approximately that of the surrounding ISM. We'll talk more about these, later on in the course.

6.4 Inflow: Spherical Accretion

Accretion flows are common in many areas of astrophysics. Galactic binary systems involving accretion are common. Cataclysmic variables, novae, and Type I supernovae involve accretion (usually non-steady) onto white dwarfs. X-ray binaries (which come in several different flavors) involve accretion (usually thought to be steady) onto neutron stars or black holes. In addition, star formation must proceed through accretion (as the outer regions of the protostar accrete onto the inner regions). Millisecond pulsars ("recycled pulsars") are often found in binary systems, and are believed to have been spun-up by accretion of mass and angular momentum from a compansion. Finally, accretion onto a massive nuclear black hole is the most likely explanation for active galactic nuclei (quasars, radio galaxies, Seyfert galaxies, and all related phenomena).

Most astrophysical accretion flows involve angular momentum, and thus we need to worry about accretion disks. We'll do that later; for now, let's consider simple spherical accretion.

6.4.1 Basic ideas

To start, we first address simple considerations of energetics and temperature. Following this, we will look at models of spherical and then disk accrestion.

Energetics. The basic consideration is that gravitational potential energy is released, at a rate $\dot{E}_g$; and can be turned into radiation with some (as yet unspecified) efficiency ε:

$$\dot{E}_g \sim \frac{GM\dot{M}}{r} \qquad \Rightarrow \qquad L = \varepsilon \frac{GM\dot{M}}{r} \tag{6.13}$$

This simple formulation is the foundation for a wide range of models (and pure wild speculation) wherever accretion is happening in astrophysics (neutron stars, star-sized black holes, supermassive black holes ..); we'll come back to it again and again.

Temperatures. What can we say, simply, about the temperature of the inflowing material? One simple limit is when the material is optically thick. In this case, the luminosity from (6.13) is re-radiated as a black body, for which we know (from thermodynamics) that the luminosity per surface area is $\sigma_{SB}T^4$, wher eσ_{SB} is the Stefan-Boltzmann constant. Equating the luminosity in (6.13) to that lost by black body radiation determines the temperature the gas will reach. This calculation finds $T \sim 10^7$K for accretion onto a solar-sized black hole (and thus we have galactic X-ray binaries); and $T \sim 10^4 - 10^5$K for accretion onto a massive $M \sim 10^9 M_\odot$ black hole (as in a galactic nucleus).

6.4.2 Spherical (Bondi) accretion

To continue, consider spherical accretion specifically. This would describe, for instance, the rate at which a compact object would accumulate matter from the general ISM; or it could describe the growth of a protostar inside a dense cloud (until angular momentum and magnetic fields become important).

We look only at a simple case of spherical accretion, that of smooth, adiabatic inflow. This simple case will be describable by the basic wind equation, (6.9), with $dv/dr < 0$ solutions. The rate of inflow, then, must be determined by conditions "at infinity", that is, in the local ISM. (6.9) has smooth, transonic, steady solutions which start at low velocity at large distances, pass smoothly through the sonic point (6.10) and become supersonic at small radii. These solutions must of course shock down somewhere close to the stellar surface (unless we have a black hole at the origin) – but if the sonic point is well outside the surface, we can assume nearly-steady flow as described by the wind equation, over most of space.

Here, we will consider adiabatic accretion; assuming the inflowing gas is tenuous and hot, and does not have time to cool as it is compresses. Rather than consider full solutions of the force equation (6.9 in this application), we will use a simpler approach to study the nature of these solutions. In particular, we want to estimate the mass inflow rate, $\dot{M} = 4\pi r^2 \rho(r) v(r)$, evaluated at some r where we know ρ and v.

For adiabatic flow, write $p = K\rho^\gamma$ and $c_s^2 = \gamma p/\rho$. From this,

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{\gamma}{\gamma-1}\frac{d}{dr}\left(\frac{p}{\rho}\right)$$

Thus, the basic momentum equation,

$$v\frac{dv}{dr} + \frac{1}{\rho}\frac{dp}{dr} = -\frac{GM}{r^2}$$

tells us that

$$\frac{1}{2}v^2 + \frac{1}{\gamma-1}c_s^2 - \frac{GM}{r} = \text{constant} \qquad (6.14)$$

We can now evaluate this constant

- at $r \to \infty$: $v = 0$ by assumption, so the constant has the value $c_{s,\infty}^2/(\gamma-1)$.
- at $r = r_s$, the sonic point: $v = c_s$, and $GM/r = 2c_s^2$; so the constant is $\frac{1}{2}c_s^2(r_s) + \frac{1}{\gamma-1}c_s^2(r_s) + 2c_s^2(r_s)$.

Equating these two, we can relate the sound speed (that is, the internal energy) at r_s to that of the ISM:

$$c_s^2(r_s) = \left(\frac{2}{5-3\gamma}\right)c_{s,\infty}^2 \qquad (6.15)$$

We can also evaluate the density at r_s, from $c_s^2 = \gamma p/\rho = K\gamma\rho^{\gamma-1}$:

$$\rho(r_s) = \rho_\infty\left(\frac{c_s(r_s)}{c_{s,\infty}}\right)^{2/(\gamma-1)} \qquad (6.16)$$

Thus, we can evalute $\dot{M}$ at r_s, and get our desired result, in terms of ISM quantities:

$$\dot{M} = 4\pi r_s^2 \rho_\infty c_{s,\infty}\left(\frac{c_s(r_s)}{c_{s,\infty}}\right)^{(\gamma+1)/(\gamma-1)} \qquad (6.17)$$

where we have used (6.9) to express r_s in terms of the stellar mass, M, and conditions at ∞. With more algebra we can show

$$\dot{M} = \pi(GM)^2\frac{\rho_\infty}{c_{s,\infty}^3}\left(\frac{c_s(r_s)}{c_{s,\infty}}\right)^{(5-3\gamma)} \qquad (6.18)$$

Thus, the important dependence is, $\dot{M} \propto M^2\rho_\infty/c_{s,\infty}^3$; the $c_s(r_s)/c_{s,\infty}$ term is just an order-unity numerical constant (from 6.15).[1]

Key points

- Why the sound speed is important; why subsonic and supersonic flows can behave differently.
- Stellar winds: the sonic point; behavior inside and outside r_s.
- What happens with $\mathbf{B}$ fields in the solar wind.
- Spherical accretion: basic energetics
- Spherical accretion: Bondi model, how $\dot{M}$ relates to conditions at ∞.

[1] You might be worried that this constant blows up when $\gamma = 5/3$, which is our favorite value for γ. Not so; with l'Hopital's rule you can show that

$$\lim_{\gamma\to 5/3}\left(\frac{2}{5-3\gamma}\right)^{(5-3\gamma)/2(\gamma-1)} \to 1$$

Try it for yourself .. it's a nice little exercise.