7 Wave propagation in plasmas

In this chapter we revisit waves in plasmas, more formally. This can be quite a mathematical topic. I'll store some of the math details in these notes, but try to highlight the physics (especially in class). If you want more detail, or background, good references are Rybicki & Lightman, *Radiative Processes in Astrophysics*; Chen, *An Introduction to Plasma Physics and Controlled Fusion*; and Choudhuri, *The Physics of Fluids and Plasmas*.

7.1 Plasma Oscillations

First, let's revisit the plasma waves, which we've seen once already. We want a more formal derivation: how do the particle motions and charge separation connect to each other?

7.1.1 Cold plasma

Start with a cold plasma – that is, ignore thermal effects; the only particle motions are those in response to the electric field. The basic equation of motion for particles of charge q is[1]

$$mn\frac{d\mathbf{v}}{dt} = mn\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = qn\mathbf{E} \tag{7.1}$$

But the charge distribution determines $\mathbf{E}$; thus we need to add two more equations,

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\,; \qquad \nabla\cdot\mathbf{E} = 4\pi\rho \tag{7.2}$$

(where ρ is the *net* charge density). For simple plasma waves, we will assume the ions don't move (because they are so heavy), and just look at the electrons.

Now, we want to look for small-amplitude, wavelike disturbances. Thus, we first *linearize* - assume all dependent variables can be broken down into (unperturbed) + (perturbed) parts, with the perturbations being small. That is:

$$n \to n_o + n_1\,; \quad \mathbf{E} \to \mathbf{E}_o + \mathbf{E}_1\,; \quad \mathbf{v} \to \mathbf{v}_o + \mathbf{v}_1 \tag{7.3}$$

We put these into (7.1) and (7.2), and sort terms by their "order in small": the zero-th order terms (n_o, etc) represent any unperturbed equilibrium state, & should cancel out; the second-order-small terms (n_1^2, etc) are small & can be dropped; so we keep just the first-order small terms. That gives us,

$$\frac{\partial n_1}{\partial t} + n_o\nabla\cdot\mathbf{v}_1 = 0\,; \quad \nabla\cdot\mathbf{E}_1 = -4\pi e n_1\,; \qquad m\frac{\partial \mathbf{v}_1}{\partial t} = -e\mathbf{E}_1 \tag{7.4}$$

These are the equations we want to solve. We make two simplifying choices here. One, we assume[2] each perturbation $f_1(\mathbf{r},t) \to \tilde{f}_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, and two, we choose the geometry: pick $\mathbf{E}_1 \parallel \mathbf{v}_1 \parallel \mathbf{k}$. Our three equations (7.4) become

$$ik\tilde{E}_1 = -4\pi e\tilde{n}_1\,; \quad -i\omega\tilde{n}_1 = -n_o ik\tilde{v}_1\,; \qquad -i\omega m\tilde{v}_1 = -e\tilde{E}_1 \tag{7.5}$$

(remember that $\tilde{n}_1$, etc, are the amplitudes of the perturbations). But now: (7.5) is a linear, homogeneous system in (n_1, E_1, v_1); it has non-trivial solutions only if the determinant of the coefficients is zero. This translates to an important condition on the frequency:

$$\omega^2 = \omega_p^2 = \frac{4\pi n_o e^2}{m} \tag{7.6}$$

Thus, we've recovered the *plasma frequency* – the frequency at which charge-separated perturbations oscillate. NOTE that these are *not* propagating waves, because ω is independent of the wavenumber k (thus the group velocity $v_g = d\omega/dk = 0$).

7.1.2 Warm plasma waves

Now, extend this to include the effects of internal energy in the plasma. That is, add a ∇p term to (7.1):

$$mn\frac{d\mathbf{v}}{dt} = mn\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = qn\mathbf{E} - \nabla p \tag{7.7}$$

Here, $p = nkT$ as usual. We'll assume an adiabatic perturbation, so that $\nabla p = (\gamma p/n)\nabla n$; and for one-dimensional motion, with one degree of freedom, we have $\gamma = 3$. Carry through the same analysis as above, with the extra pressure term; and you find

$$\omega^2 = \omega_p^2 + k^2\frac{3kT}{m} = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2 \tag{7.8}$$

[1] We can think of the LHS as the "total" time rate of change, as seen by the particle moving with velocity $\mathbf{v}$; you can also compare the LHS of the force equation for a fluid, eqn (4.4).

[2] Why can we do this? Think about Fourier analysis – any perturbation can be expressed as the sum of its Fourier components. We can get away with this because equations (7.4) are *linear* in the dependent variables.

where the (1D) thermal speed is $v_{th}^2 = 2kT/m$. These are now propagating waves; they have a nonzero group velocity, $v_g = d\omega/dk$.

7.1.3 Damping: collisional

Plasma waves are subject to two types of damping. One – which you'd expect – is *collisional*. If the electrons collide with other charges as they respond to the wave motion, the wave energy will of course go to heating. The equation of motion for the electrons gains a collisional term:

$$\begin{aligned} mn\frac{d\mathbf{v}}{dt} &= mn\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] \\ &= qn\mathbf{E} - \nu_{coll} nm\mathbf{v} \end{aligned} \tag{7.9}$$

where ν_{coll} is the collision rate of an individual charge. (This can be due to Coulomb collisions in a fully ionized plasma, or electron-neutral collisions in a partly ionized one.) It's easy to show that the effect of this is to add an imaginary component to the wave frequency, $\omega = \omega_R + i\omega_I$; and thus our waves are exponentially damped: $E_1 \propto e^{i\omega t} \propto e^{-\omega_I t}$ (the sign of ω_I determines whether the wave grows or is damped; physically for simple collisions it must be damped, right?)

7.1.4 Damping: collisionless

Another, less intuitive, process is *collisionless* damping, also called Landau damping. The derivations of this are usually quite mathematical — but we really need a physical understanding more, so that's all I'll do in these notes. Think about a warm-plasma wave, which travels at a phase speed $v_{ph} = \omega/k$; if a charged particle has v_x exactly equal to v_{ph}, it will be in "resonance" with the wave – it will see exactly the same phase of the wave as it moves along. Now, consider another particle moving at a velocity v_x which is very close to v_{ph}. This particle will see almost a constant phase of the wave; in fact, it will be "captured" by the wave, and will "ride along" with a crest or trough of the wave. If the particle is slightly slower – if $v_x \lesssim v_{ph}\omega/k$ – it will be accelerated up to v_{ph} – draining a wee bit of the wave energy in the process. Conversely, if $v_x \gtrsim v_{ph}$, the particle will be decelerated as it is "captured", thus giving a bit of energy up to the wave.

Think, then, about the net effect of this particle-wave interaction in a plasma. Two cases are possible (refer to class notes for a cartoon).

- If the distribution function $f(v_x)$ of the plasma is "normal", say a Maxwellian as you'd expect from thermal equilibrium, then it has negative slope: $df/dv_x < 0$ at $v_x \simeq v_{ph}$. Thus, there will be slightly more particles at $v_x < v_{ph}$ then at $v_x > v_{ph}$. That means that more particles will gain energy from the wave than will lose energy to the wave: the wave is *damped*, without needing any particle-particle collisions. This is *Landau damping*.
- What if the distribution function has a positive slope – $df/dv_x > 0$ at $v_x = v_{ph}$? Well, the opposite will occur: the wave will gain energy at the expense of the plasma. But how do you get such a distribution function? Think about two plasmas impinging on one another – or trying to send a "beam" or "stream" of charges through a thermalized background plasma. This situation is unstable: the energy of the relative motion between the plasma streams is fed into waves of the appropriate phase velocity. This is called the *two-stream instability*. Because of this instability, one plasma trying to penetrate another cannot do so – instead a background of turbulent plasma waves is generated.

Isn't this last result curious? You might naively think that two low-density plasmas could interpenetrate each other without any problem, as long as Coulomb collisions are unimportant. But that's not the case: the two-stream instability, which also arises from long-range Coulomb interactions, tries to keep the plasmas separate.

7.2 EM wave propagation: B = 0

In the last section we considered intrinsic oscillations in the plasma. Now we change the setup: send in an EM wave (and let's stick to cold plasma). This means we'll have two important changes in our analysis – (i) we need to include $\mathbf{B}$ effects (at least the $\mathbf{B}$ field of the wave), and (ii) now we take $\mathbf{E}_1, \mathbf{B}_1 \perp \mathbf{k}$ (EM waves are transverse, after all). We want to know how the wave propagates and how it affects the plasma; and – for astrophysics – how to interpret observations in terms of what's going on in the plasma.

7.2.1 Basic: the dispersion relation

Start here with the full Maxwell set:

$$\begin{aligned} \nabla\cdot\mathbf{E} = 4\pi\rho\,; \quad \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0\,; \quad \nabla\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \end{aligned} \tag{7.10}$$

We are still imposing plane waves, $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$; putting these into Maxwell, we get

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{E} = 4\pi\rho\,; &\quad i\mathbf{k}\times\mathbf{E} = i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\cdot\mathbf{B} = 0\,; &\quad i\mathbf{k}\times\mathbf{B} = -i\frac{\omega}{c}\mathbf{E} + \frac{4\pi}{c}\mathbf{j} \end{aligned} \tag{7.11}$$

Now work on the source terms; we know $\mathbf{j} = -ne\mathbf{v}$ (again, only the electons move). We still have the simple equation of motion, (7.1), and its linearized/wave mode form, $im\omega\mathbf{v} = e\mathbf{E}$. Thus, we can write[3]

$$\mathbf{j} = \sigma\mathbf{E}\,; \qquad \sigma = \frac{ine^2}{m\omega} \tag{7.12}$$

We need to close the system with conservation of charge:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \quad\Rightarrow\quad i\omega\rho = i\mathbf{k}\cdot\mathbf{j} \tag{7.13}$$

Now, put these back in (7.11) and collect terms:

$$\begin{aligned} i\mathbf{k}\cdot(\epsilon\mathbf{E}) = 0\,; &\qquad i\mathbf{k}\times\mathbf{E} = i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\cdot\mathbf{B} = 0\,; &\quad i\mathbf{k}\times\mathbf{B} = -i\frac{\omega}{c}(\epsilon\mathbf{E}) \end{aligned} \tag{7.14}$$

Here, we've identified the *dielectric factor*

$$\epsilon = 1 - \frac{4\pi\sigma}{i\omega} = 1 - \frac{\omega_p^2}{\omega^2} \tag{7.15}$$

(Comment: many books call this the dielectric constant, in analogy with EM propagation in dielectric materials – but here ϵ is a function of frequency, so it's hardly constant). By analogy with what you've seen of EM waves elsewhere, once we have the equations in the form (7.14), we know the wave solution is $\omega^2\epsilon = c^2k^2$. Thus, our dispersion relation for the wave propagating in the plasma is

$$\omega^2 = k^2c^2 + \omega_p^2 \tag{7.16}$$

and we want to look at simple applications of this.

7.2.2 Applications and extensions

Several important phenomena deserve mention here.

• **Plasma cutoff**. It's clear from (7.16) that real k's are allowed only if $\omega > \omega_p$: *waves below the plasma frequency do not propagate*. Why is this? Mathematically, think about k being imaginary: the wave form $e^{ikx} \to e^{-|k|x}$, that is the wave damps exponentially. Physically, think about how the plasma responds to an incoming EM wave of frequency ω. For $\omega \ll \omega_p$, the plasma charges can easily "move up and down" with the wave – that is they can easily absorb the wave energy. But for $\omega \gg \omega_p$, the charges can't move fast enough (think about driving an oscillator well above its natural frequency) – so the wave happily propagates through the plasma.

• **Plasma dispersion**. Because $\omega(k)$ in (7.16) is dispersive, waves at different frequencies move at different phase speeds. This is important when you're looking at an astrophysical object – such as a pulsar - which emits very short pulses. The arrival time of such a pulse at earth, from a distance D away, depends on the frequency. Remembering that $v_g = d\omega/dk$, the arrival time of a pulse is (switching to $\nu = \omega/2\pi$, to connect to observations)

$$t_p(\nu) = \int_0^D \frac{ds}{v_g} \simeq \int_0^D \left(1 + \frac{\nu_p^2}{\nu^2}\right)^{1/2} \frac{ds}{c} \tag{7.17}$$

(You should note that I've assumed $\nu \gg \nu_p = \omega_p/2\pi$ here.) The frequency-dependent term in (7.17) is usually scaled as

$$\int_0^D \frac{ne^2}{\pi mc}\frac{1}{\nu^2}ds \simeq 4.15\times10^{15}\frac{(\mathrm{DM})}{\nu^2}\ \text{sec} \tag{7.18}$$

where the *dispersion measure*, $\mathrm{DM} = \int_0^D n ds$, is measured in cm^{-3}pc, and the frequency ν is in Hz.

• **collisional damping**. As with plasma waves, EM waves also can damp out in a plasma. Equation (7.9) is still the starting point. Working this through for our EM wave, we find the dispersion relation is

$$\omega^2 = \frac{\omega_p^2\omega}{\omega + i\nu_{coll}} + c^2k^2 \tag{7.19}$$

This looks rough; it's a cubic equation in ω. However things simplify if the damping rate is small compared to the wave frequency (or its real part). The solutions to (7.19) must be complex. To approach them, we can either hold ω real and find complex k, or vice versa. The first approach corresponds to driving a plasma with an incoming wave of fixed ω; the imaginary part of k corresponds to the spatial damping (wave absorption by the plasma). Alternatively, we can hold k fixed and

[3] What does an imaginary conductivity, σ, mean? Remember that $e^{-x} = \cos x + i\sin x$; what does an imaginary σ tell us about the phase relation between $\mathbf{j}$ and $\mathbf{E}$?

look for the imaginary part of ω – some authors prefer this approach. In either case: (7.19) is a cubic, and potentially intimidating; but solutions simplify if the damping is weak, $\nu_{coll} \ll \omega$ (or $\nu_{coll} \ll \Re(\omega)$ if you prefer).

7.3 EM wave propagation: finite B

Well, this has been so much fun, let's do it again. Now add a magnetic field $B = B\hat{\mathbf{z}}$ to the background plasma, and consider waves propagating along the field: $\mathbf{k} \parallel \mathbf{B}$ (this is the only simple geometry!).

7.3.1 Basic: the dispersion relation

We'll only do the outline here, you have the details above. The equation of motion for the electrons now is

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{7.20}$$

(caution: here $\mathbf{E}$ is the wave field, but $\mathbf{B}$ is the background field. I'm assuming the wave B field is small & ignoring it). Now: for the incoming EM wave we choose circular polarization:

$$\mathbf{E} = Ee^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) \tag{7.21}$$

where the $\pm$ signs pick out RH or LH circular polarization. (Why? think about the response of the plasma charges – the electrons have a preferred sense of gyromotion – so we might expect RH and LH circularly polarized waves to have different phase speeds – as the electrons might help or hinder them.) Carry out the same type of analysis – the electron response is

$$\mathbf{v} = \frac{-ie}{m(\omega \pm \Omega)}\mathbf{E} \tag{7.22}$$

where $\Omega = eB/mc$ is the gyrofrequency. The dielectric factor becomes

$$\epsilon_{R,L} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \Omega)} \tag{7.23}$$

Thus, RH and LH waves do propagate at different phase speeds in the medium. The important application of this is to the angle of linear polarization of an incoming wave. We also get the dispersion relation, directly, from this, from the definition of ϵ:

$$\omega^2 = c^2k^2 + \frac{\omega_p^2}{(1 \pm \Omega/\omega)} \tag{7.24}$$

7.3.2 Applications and extensions

Once again we have some important applications.

• **"Plasma" cutoffs** also occur here, they're just a bit more complicated. We again want the frequency, from (7.23), at which $k = 0$ – that's the transition between real (propagating) and imaginary (damped) waves. The answer depends on the sign choice in (7.23) (that is, on whether the waves are RH or LH polarized). We get the critical frequencies for transmission:

$$\omega_{R,L} = \frac{1}{2}\left[\pm\Omega + \left(\Omega^2 + 4\omega_p^2\right)^{1/2}\right] \tag{7.25}$$

Thus, LH waves only propagate for $\omega > \omega_L$. However, RH waves propagate above ω_R and also below Ω (you check the algebra: there are two domains of $k^2 > 0$ for this polarization).

• **Faraday rotation** is one of the most important astrophysical methods for measuring magnetic fields. Remember that the phase of a wave, which has travelled a distance D, is $\phi = \int_0^D kds$. Thus the phase difference between R,L waves is

$$\Delta\phi = \int_0^D (k_R - k_L)ds\ ; \quad k_{R,L} = \frac{\omega}{c}\sqrt{\epsilon_{R,L}} \tag{7.26}$$

Now, because a linearly polarized wave can be written as the sum of RH and LH circularly polarized waves, the angle χ by which the polarization is rotated is 1/2 of the phase difference: $\chi = (\Delta\phi)/2$.

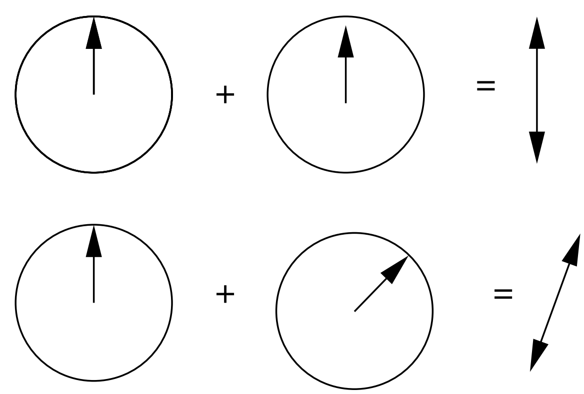

Figure 7.1 Decomposition of linear polarization into components of right and left circular polarization. Top, RC and LC in phase; bottom, phase shift $\Delta\phi$ between RC and LC (as due to Faraday rotation), rotates the plane of polarzation by $\chi = \Delta\phi/2$. Following Rybicki & Lightman figure 8.1.

But now, going back to (7.21) and assuming that our observed wave frequency is well above the plasma and cyclotron frequencies of the plasma, we can expand k as

$$k_{R,L} \simeq \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}\left(1 \mp \frac{\Omega}{\omega}\right)\right] \tag{7.27}$$

Putting this back in (7.22) and doing more algebra, we get

$$\chi = \frac{\lambda^2}{2\pi}\frac{e^3}{m^2c^4}\int_0^D n\mathbf{B}\cdot d\mathbf{s} \tag{7.28}$$

(Note that I've gone to the more general $\mathbf{B}\cdot d\mathbf{s}$ in the integrand; for a general direction of wave propagation relative to $\mathbf{B}$, it turns out to be the component of $\mathbf{B}$ along the line of sight that matters.) This result, (7.27), is generally scaled as $\chi = \lambda^2(\mathrm{RM})$, where the *rotation measure* is

$$\mathrm{RM} = \frac{1}{2\pi}\frac{e^3}{m^2c^4}\int_0^D n\mathbf{B}\cdot d\mathbf{s} \tag{7.29}$$

A convenient numerical scaling for rotation measure is

$$(\mathrm{RM}) \simeq 810\int_0^D n\mathbf{B}\cdot d\mathbf{s}$$

for n in cm^{-3}, B in μG, s, D in kpc, and RM in rad/m^2.

Key (physical) points

• Wave analyses (e.g. plasma waves): the mathematical attack (start with basic equations; linearize; assume $e^{i(kx-\omega t)}$ single-frequency solutions; do the math and see what you get.

• Plasma waves: when do they propagate, how fast, how do they damp?

• Collisional dissipation (of various types of waves)

• EM waves in plasma: plasma dispersion effects

• EM waves in plasma: $\mathbf{B} \neq 0$ effects, Faraday rotation