4 Bremsstrahlung radiation

Bremsstrahlung arises when a free electron is accelerated in the field of an ion – hence the English name "free-free," representing the transition between two unbound electronic states. The German name "braking radiation" refers to the acceleration of the electron.

4.1 Some basic tools

Before we start bremsstrahlung *per se*, we need to introduce two important general tools used for general analysis of astrophysical radiation.

4.1.1 Power; Larmor formula

You probably remember that if you shake an electron, it will radiate E&M waves. We want to connect the total power in the E&M radiation to "how hard the electron was shaken".

The formal result is that a charge e, which feels an acceleration $\mathbf{a}(t)$, radiates a power (erg s^{-1}) given by

$$\text{cgs} \; : \qquad P(t) = \frac{2}{3}\frac{e^2}{c^3}|\mathbf{a}(t)|^2 \tag{4.1}$$

Note this is cgs.[1]

To derive this, you need to work out the $\mathbf{E}$ and $\mathbf{B}$ fields which are produced by the accelerated charge; then fold them together into the Poynting flux, $\mathbf{S} = c\mathbf{E}\times\mathbf{B}/4\pi$.[2] That gives us the energy per unit area per unit time carried away from the particle by the fields, *i.e.* the radiated power. Griffiths has a good derivation in chapter 11; Rybicki & Lightman have a more terse derivation in chapter 3. You should note that this formula holds for non-relativistic motion; we'll extend it to the relativistic case, later.

4.1.2 Spectrum: Fourier analysis

The other basic tool we need is the **spectrum** of the radiation. When our electron is shaken, it emits a pulse of radiation, which has a finite duration. This pulse is a wave packet – a superposition of E&M waves of various frequencies. The pulse width in time, Δt, is related to the range of frequencies in the packet, $\delta\nu$, by the usual uncertainty principle: $\Delta t\Delta\nu \sim O(1)$ (as in Figure 4.1). The amplitude of frequency component ν gives what we'll identify as the radiation spectrum.

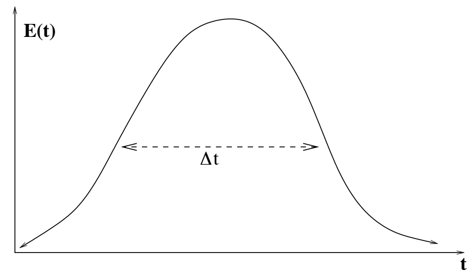

Figure 4.1 Remembering the wave content of a wave packet. The time duration of this packet $\sim \Delta t$ (note this is estimated "by eye" here); it contains frequencies $\lesssim 1/\Delta t$.

To get there formally, we use the Fourier transform ("FT"). I take this from chapter 2 of Rybicki & Lightman. Think about our pulse of radiation; let the electric field in the pulse have some time behavior, $\mathbf{E}(t)$. We can, of course, consider the Fourier transform of this, and its inverse (I'll drop vectors to simplify the notation):

$$\begin{aligned} \hat{E}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt \\ \Leftrightarrow \quad E(t) &= \int_{-\infty}^{\infty} \hat{E}(\omega)e^{-i\omega t}d\omega \end{aligned} \tag{4.2}$$

Two FT facts will be useful. First, because $E(t)$ is real, we know that

$$\hat{E}(-\omega) = \hat{E}^*(\omega)\;;\;\; |\hat{E}(-\omega)|^2 = |\hat{E}(\omega)|^2 \tag{4.3}$$

(that is, the negative frequencies contain no new information). Second, a general result from Fourier transforms (Parseval's theorem) tells us that

$$\begin{aligned} \int_{-\infty}^{\infty} E(t)^2 dt &= 2\pi\int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega \\ &= 4\pi\int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega \end{aligned} \tag{4.4}$$

(I've used 4.3 in the last step).

Now: the total energy per unit area emitted in the radiation pulse is

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E(t)^2 dt \tag{4.5}$$

(to see this, start with the Poynting flux, and remember that $E = B$ for an EM wave in cgs). Now by (4.4), we can rewrite this as

$$\frac{dW}{dA} = c\int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega \tag{4.6}$$

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[1] In SI, the formula is

$$\text{SI} \; : \qquad P(t) = \frac{\mu_o}{6\pi}\frac{e^2}{c}|\mathbf{a}(t)|^2$$

[2] or, $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_o$ in SI.

So: we're just about there. What we do, essentially, is think of the integral in (4.6) as an integral over the frequency spectrum of the radiation pulse: that is, $c|\hat{E}(\omega)|^2$ measures the energy in the pulse "at ω".[3]

Thus: looking back to (4.1), we can connect this formalism to the Larmor result. Compare (4.1) to (4.6): they both described the total energy within the pulse. So, think about the Fourier transform of some component (x, y or z) of the acceleration:

$$a_i(t) = \int_{-\infty}^{\infty} \hat{a}_i(\omega) e^{-i\omega t} d\omega \tag{4.7}$$

We can thus identify

$$P(\omega) = \frac{8\pi}{3}\frac{e^2}{c^3}|\hat{a}(\omega)|^2 \tag{4.8}$$

with what we want, namely, the *spectrum* of the radiation. The 4π difference between the frequency-space definition here and the real-time definition (4.1) arises in the Fourier transform (4.4).

4.2 Bremsstrahlung I: single particle

We can now apply this to radiation from an electron-ion collision, using the usual geometry (which you remember from chapter 3 of our Phys 425 notes).

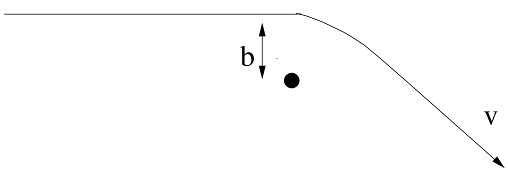

Figure 4.2 The usual geometry, as an electron is deflected by an ion. The impact parameter is b; let the electron move in the x direction to start, with velocity v.

The two components of acceleration are

$$\begin{aligned} a_x &= \frac{e^2 vt}{m(b^2+v^2t^2)^{3/2}} \\ a_z &= \frac{e^2 b}{m(b^2+v^2t^2)^{3/2}} \end{aligned} \tag{4.9}$$

where we have used the impact parameter, b, have set the origin of time at the time of closest approach, and have assumed the particle suffers only a small deflection, so that the $\mathbf{v}$ does not change by much. The radiated spectrum will depend on the FT of this acceleration. Without doing this out algebraically, we can predict the answer, using what we know about Fourier transforms. In particular, we note that

(i) The z-component of the acceleration will be the dominant factor over the course of the encounter because it does not go to zero at closest approach.

(ii) Since $a(t)$ is large only when the two particles are close together – just as we argued in the Coulomb collision discussion – $P(t)$ will be significant only for times $\lesssim 2b/v$.

(iii) Therefore, we expect the spectrum will be dominated by the FT of a_z, and that the FT will have power at frequencies $\omega \lesssim v/2b$.

Doing the actual work, Longair gives the result for both parallel and perpendicular acceleration:

$$\begin{aligned} \hat{a}_x(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^2 vt}{m_e}\frac{e^{i\omega t}}{(b^2+v^2t^2)^{3/2}} dt \\ &= \frac{1}{2\pi}\frac{e^2}{m_e b v}\frac{2\omega b}{v} i K_0\left(\frac{\omega b}{v}\right) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \hat{a}_z(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^2 b}{m_e}\frac{e^{i\omega t}}{(b^2+v^2t^2)^{3/2}} dt \\ &= \frac{1}{2\pi}\frac{e^2}{m_e b v}\frac{2\omega b}{v} K_1\left(\frac{\omega b}{v}\right) \end{aligned} \tag{4.11}$$

where $K_o(\omega b/v)$ and $K_1(\omega b/v)$ are modified Bessel functions. We can find analytic forms for K_1 and K_o in the limits of large and small arguments:

$$\begin{aligned} K_0(x) &\to -\ln x & x \ll 1 \\ &\to \sqrt{\frac{\pi}{2x}} e^{-x} & x \gg 1 \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} K_1(x) &\to \frac{1}{x} & x \ll 1 \\ &\to \sqrt{\frac{\pi}{2x}} e^{-x} & x \gg 1 \end{aligned} \tag{4.13}$$

From this, we can add both acceleration terms (squared) to get the radiated spectrum:

$$P(\omega) = \frac{8\pi e^2}{3c^3}\left[|\hat{a}_x(\omega)|^2 + |\hat{a}_z(\omega)|^2\right] \tag{4.14}$$

[3] There is an important technical detail here. The expression (energy/area) in (4.5) or (4.6) is *not* per unit time, rather it's integrated over the pulse. If we tried to do this argument "per dt" and "per $d\omega$", we'd violate the uncertainty relation between ω and t in the wave packet. However, if the pulses repeat frequency, one can formally take limits and get the same result ... *cf.* RL for details here.

Using the analytic expressions for the modified Bessel functions, we find the limiting forms for the spectrum radiated in a single-particle encounter:

$$\begin{aligned} P(\omega) &\simeq \frac{8}{3\pi}\frac{e^6}{m_e^2 c^3 v^2 b^2} & \omega \ll \frac{v}{2b} \\ P(\omega) &\simeq \frac{8}{3\pi}\frac{e^6}{m_e^2 c^3 v^2 b^2} e^{-\omega b/v} & \omega \gg \frac{v}{2b} \end{aligned} \tag{4.15}$$

You should note that the exponential cutoff in the second equation, here, means there is effectively no radiation above $\omega \sim v/b$ — which is consistent with what we expect from the duration of the wave packet.

The low frequency limit of (4.15) is worth commenting on. $P(\omega) \to$ constant as $\omega \to 0$, and is well behaved. However, the *photon* emissivity, $P(\omega)/\hbar\omega$ diverges as $1/\omega$ as $\omega \to 0$. This is a well-known problem in both classical and quantum electrodynamics, known as the "infrared divergence." This is essentially an artifact of our derivations, rather than a problem with the physics (for instance, see Jauch and Rohrlich, *The Theory of Photons and Electrons*). Here, we will simply note that the *energy* lost is finite, and that self-absorption in any finite system (see below) will keep us from ever seeing this divergence anyway. We will, therefore, carry on happily.

4.3 Bremsstrahlung II: from a plasma

Now, we want to extend this to consider a particle in a plasma, and to take all of its collisions into account. We did this last term, when we derived Coulomb scattering – we had to take all impact parameters into account. We'll do essentially the same thing here.

First, consider the range of impact parameters that one particle encounters. Since the number of ions that one electron, at velocity v, sees per second at impact parameter b is $2\pi n_i v b db$, we can find the total radiated spectrum from that electron,

$$\begin{aligned} P(\omega, v) &= \int_{b_{\min}}^{b_{max}} P(\omega, v, b) 2\pi n_i v b \, db \\ &= \int_{b_{min}}^{b_{max}} \frac{8}{3\pi}\frac{e^6}{m_e^2 b^2 v^2 c^3} n_i v 2\pi b \, db \\ &= \frac{16}{3}\frac{e^6 n_i}{m_e^2 v c^3} \ln\left(\frac{b_{max}}{b_{min}}\right) \end{aligned} \tag{4.16}$$

Again, the range of impact parameters must be chosen with some physics in mind. And again, luckily, our choice only affects the answer logarithmically. Typical choices are $b_{min} \sim e^2/m_e v^2$, and $b_{max} \sim v/\omega$ or $\sim \hbar/2m_e v$.

Next, we use this to find the total energy loss rate for one particle. We do this by integrating $P(\omega, v)$ over all frequencies. Since $P(\omega, v)$ is only a weak function of ω (it appears only in $\ln(b_{max}/b_{min})$), up to the frequency cutoff $\omega_{max} \simeq m_e v^2/\hbar$ (which is the highest photon frequency we can expect, from simple energy conservation), we have

$$\begin{aligned} P(v) &= \int_{\omega_{min}}^{\omega_{max}} P(\omega, v) d\omega \\ &= \frac{16}{3}\frac{e^6}{c^3 m_e \hbar} \ln\left(\frac{b_{max}}{b_{min}}\right) n_i v \end{aligned} \tag{4.17}$$

This has the functional form $P(E) \propto E^{1/2}$.

Returning to the single particle spectrum, (4.15), we can now integrate over all particles in the plasma, to get the total emissivity from that plasma. We need to know the distribution of electron speeds, and we will assume a Maxwell-Boltzmann distribution. We also switch from ω to $\nu = \omega/2\pi$, to connect with observations; and we derive $j_{ff}(\nu)$, the emissivity per steradian, to connect with the radiative transfer applications, above. (That means simply a 4π factor, since the single particle emission is essentially isotropic). Thus,

$$j_{ff}(\nu) = \frac{1}{4\pi}\int_0^\infty P(\omega, v) f(v) dv \tag{4.18}$$

with $f(v)$ assumed to be the Maxwellian of (3.9), normalized to n_e. This gives us

$$\begin{aligned} j_{ff}(\nu) = \frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2} &\frac{e^6}{m_e^{3/2} c^3} \\ &\times \frac{n_e n_i}{(k_B T)^{1/2}} g_{ff}(\nu, T) e^{-h\nu/k_B T} \end{aligned} \tag{4.19}$$

Numerically, with everything in cgs units, this is

$$\begin{aligned} j_{ff}(\nu) =& 5.44 \times 10^{-39} g_{ff}(\nu, T) \frac{n_e n_i}{T^{1/2}} e^{-h\nu/k_B T} \\ & \text{erg s}^{-1}\text{cm}^{-3}\text{Hz}^{-1}\text{str}^{-1} \end{aligned} \tag{4.20}$$

In this expression, we have implicitly defined the Gaunt factor, $g_{ff}(\nu, T)$. It arises from the velocity dependence of $\ln(b_{max}/b_{min})$, inside the integral in (4.18): the essential part of the integral is

$$\begin{aligned} &\int \frac{1}{v} f(v) \ln\left(\frac{b_{max}(v)}{b_{min}(v)}\right) \\ &\qquad \to \langle\frac{1}{v}\rangle \langle \ln\left(\frac{b_{max}(v)}{b_{min}(v)}\right)\rangle \end{aligned} \tag{4.21}$$

We see that the $\langle 1/v \rangle$ becomes the $(k_B T)^{-1/2}$ term; the mean of the logarithmic factor becomes $g_{ff}(\nu, t)$, the Gaunt factor. Note that both b_{max} and b_{min} might be functions of ν and of v. As with Coulomb scattering, different expressions, corresponding to different choices of b_{max} and/or b_{min}, are used in different situations. A couple of common cases are, first, in the radio range, with $h\nu \ll k_B T$:

$$\begin{aligned} g_{ff}(\nu, T) &\simeq \frac{\sqrt{3}}{\pi} \ln \left(\frac{2}{\pi} \frac{(k_B T)^{3/2}}{e^2 m_e^{1/2} \nu} \right) \\ &\simeq 10 \left(1.0 + 0.1 \log \frac{T^{3/2}}{\nu} \right) \end{aligned} \tag{4.22}$$

Second, in the X-ray range, with $h\nu \lesssim k_B T$, people use

$$g_{ff}(\nu, T) \simeq \frac{\sqrt{3}}{\pi} \ln \left(\frac{k_B T}{h\nu} \right) \tag{4.23}$$

Finally, the total emissivity of the plasma can be found, by integrating $j_{ff}(\nu)$ over all frequencies and all solid angles. This is

$$\begin{aligned} \varepsilon_{ff} &= 4\pi \int_0^\infty j_{ff}(\nu) d\nu \\ &= \left(\frac{2\pi k_B}{3m_e} \right)^{1/2} \frac{32\pi e^6}{3hm_e c^3} n_e n_i \langle g_{ff} \rangle T^{1/2} \\ &\simeq 1.4 \times 10^{-27} n_e n_i \langle g_{ff} \rangle T^{1/2} \quad \text{erg cm}^{-3}\text{s}^{-1} \end{aligned} \tag{4.24}$$

where $\langle g_{ff} \rangle$ is the mean Gaunt factor, averaged over frequency.

We are also interested in free-free absorption. This is the inverse of the emission process; a free electron absorbs a photon (the ion must be there, as well, to conserve momentum and energy at the same time). For absorption by a Maxwellian plasma, for which we have just derived the emissivity $j_{ff}(\nu)$, we can get the absorption coefficient, $\kappa_{ff}(\nu)$, immediately from Kirchoff's law (3.34):

$$\begin{aligned} \kappa_{ff}(\nu) &= \frac{4}{3\pi} \left(\frac{2\pi}{3} \right)^{1/2} \frac{e^6 n_e n_i g_{ff}(\nu, T)}{m_e^{3/2} c (k_B T)^{3/2} \nu^2} \\ &\simeq 0.018 g_{ff}(\nu, T) \frac{n_e n_i}{T^{3/2} \nu^2} \quad \text{cm}^{-1} \end{aligned} \tag{4.25}$$

This expression is valid in the Rayleigh-Jeans limit (3.22); and the second expression is all in cgs units). You should note that this expression also contains the Gaunt factor. The most common use of free-free absorption is in the radio range (given the $1/\nu^2$ form), and a commonly used form of the absorption, which includes a particular expression for the Gaunt factor, is

$$\kappa_{ff}(\nu) \simeq 0.08235 \frac{n_e n_i}{T^{1.35} \nu_{GHz}^{2.1}} \quad \text{pc}^{-1} \tag{4.26}$$

Note the oddball units: ν is in GHz; κ is in inverse pc(!); but n_e, n_i and T are still in cgs.[4] Avrett, *Frontiers of Astrophysics*, says this is good for $0.1 < \nu < 50$ GHz, and for $6000 < T < 18,000$ K (which describes HII regions and much of the warm, ionized ISM, for instance).

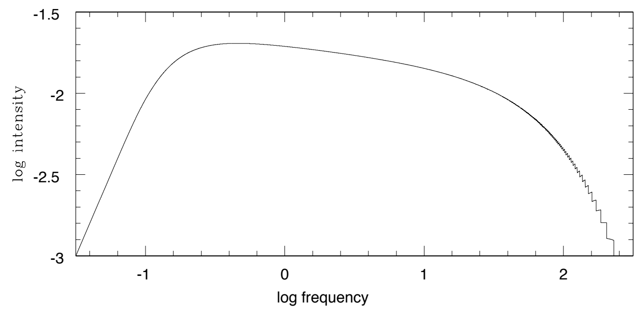

Figure 4.3 Illustrating the full bremsstrahlung spectrum we expect from a source which has a finite size, and thus a finite optical depth. Note, the jagged appearance of the high-frequency exponential is the fault of my simple plotting program, not the physics.

Finally: let's connect this to bremsstrahlung emission from a finite object. Think, for instance, about a galactic HII region (such as the Orion nebula). Typical temperatures are $T \simeq 10^4$ K, and typical densities might be $n \sim 100$ cm^{-3}; the size might be on the order of a pc. We can estimate $\tau_{ff}(\nu) = \kappa_{ff}(\nu)L$, and we find $\tau_{ff}(\nu) \simeq 1$ for frequencies in the low radio range (a fraction of a GHz, say). Below this frequency the source will be optically thick, and the emergent intensity will obey $I_\nu = B_\nu(T) \propto \nu^2$, from the Rayleigh-Jeans limit of the Planck function. At higher frequencies, the source is optically thin, and the emergent intensity has the same frequency dependence as the fundamental emissivity: $I_\nu \propto j_{ff,\nu}$. Thus, the intensity will be approximately constant at higher frequencies. At very high frequencies, $h\nu \sim k_B T$ (that is, tens of eV $\rightarrow 10^{16}$ Hz or so), the exponential cutoff will appear. Figure 4.3 sketches this behavior.

[4] That's astronomers for you ..

References

I'm mostly following a fairly nice discussion given in

• Longair (*High Energy Astrophysics, Vol II*);

and pulling some of the discussion on foundations from

• Rybicki & Lightman, *Radiative Processes in Astrophysics*

Key points (for chapter 4)

• Total power (Larmor formula);

• Wave content $\leftrightarrow$ radiation spectrum; "seat-of-the-pants" FT analysis.

• Bremsstrahlung: basic physical picture, single particle

• Bremsstrahlung: from a plasma; intrinsic spectrum and power

• Bremsstrahlung: emissivity, absorption coefficients, total spectrum