9 Synchrotron radiation

A single particle, undergoing gyromotion in a magnetic field, is of course accelerated; and thus it will radiate. When the particle is subrelativistic, this process is called cyclotron radiation; when the particle is relativistic, the process is called synchrotron radiation. We have already studied bremsstrahlung, a fundamental continuous radiation mechanism which is important for thermal astrophysical plasmas. Synchrotron radiation is the other common and important continuous emission process. It is imortant for magnetized astrophysical plasmas which contain a significant component of relativistic electrons (or, yes, positrons) – commonly called a "nonthermal" plasma.

9.1 Total power

We start with the total power radiated by the particle, over all frequencies. Let the particle have acceleration $\mathbf{a}$, with components $a_\parallel$ along its direction of motion ($\mathbf{v}$), and $a_\perp$ across the direction of motion (NOT the direction of the magnetic field, here). In the particle's rest frame, the total power radiated as a function of time is

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}|\mathbf{a}(t)|^2 \qquad (9.1)$$

In the observer's frame, this becomes

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right) \qquad (9.2)$$

Now, for gyromotion, we have $a_\parallel = 0$ and $a_\perp = eBc\beta_\perp/\gamma mc$. Let the particle have pitch angle θ – so that $\beta_\perp = \beta\sin\theta$. Then, the net power seen by the observer, per particle at energy γ and pitch angle θ, is

$$P_{sy} = \frac{2}{3}\frac{e^4B^2}{m^2c^3}\gamma^2\sin^2\theta \qquad (9.3)$$

You should note that this expression assumes $\gamma \gg 1$. From an ensemble of particles, with an isotropic distribution of pitch angles, we note that $\langle\sin^2\theta\rangle = 2/3$; thus the average single-particle power is

$$\langle P_{sy}\rangle = \frac{4}{9}\frac{e^4B^2}{m^2c^3}\gamma^2 \qquad (9.4)$$

Finally, we note that (9.4) can be rewritten in terms of the magnetic energy density, $u_B = B^2/8\pi$, and the Thompson scattering cross section, $\sigma_T = (8\pi/3)(e^2/m_ec^2)^2$, as

$$\langle P_{sy}\rangle = \frac{4}{3}c\sigma_T\gamma^2 u_B\ . \qquad (9.5)$$

9.2 Single particle spectrum

From the discussion of bremsstrahlung, recall: the power as a function of time, $P(t)$, reflects the time dependence of the acceleration, $a(t)$; the distribution of this radiation over frequency – the spectrum – reflects the power (Fourier) spectrum of $a^2(t)$. In the synchrotron case, the *observed* radiation varies with time due to relativistic beaming effects, in addition to the fundamental gyromotion. The power spectrum of the observed $P(t)$ turns out to contain power at frequencies much higher than $\Omega/2\pi$; this determines the observed synchrotron emission spectrum.[1]

• **First, guesstimate the answer.** The important fact here is that the radiation from a relativistic particle is strongly beamed in the direction of the particle's motion. A nonrelativistic particle radiates in a dipole pattern, with the dipole axis aligned with the acceleration $\mathbf{a}$. However, for a relativistic particle, this dipole is squeezed into a narrow forward cone, aligned with the motion ($\mathbf{v}$) and with opening angle $\simeq 1/\gamma$.

So, think about a relativistic particle in gyromotion. You – as an observer – only see the particle's radiation once per orbit, when the particle's velocity is within $1/\gamma$ of your line of sight. Say that the beam stays within your line of sight for a time interval, Δt^{obs}. You will then see a very narrow pulse, of duration, Δt^{obs}, once every gyroperiod. From our experience with Fourier transforms, we know that most of the radiated power must appear at a frequency $\sim 1/\Delta t^{obs}$, rather than just at Ω (as you might naively guess if you didn't think about the beaming effects).

We can estimate Δt^{obs} from geometry. Let the part of the particle's orbit for which you see the radiation have length $\Delta s = a\Delta\phi$, where a is the radius of curvature of the path. We can find a from the equation of motion:

$$\frac{\Delta v}{\Delta t} = v^2\frac{\Delta\phi}{\Delta s} = \frac{e}{\gamma mc}vB\sin\theta \qquad (9.6)$$

so that $a = \Delta s/\Delta\phi = \gamma v/\Omega_o\sin\theta$. Since $\Delta\phi = 2/\gamma$, we have $\Delta s = 2a/\gamma = 2v/\Omega_o\sin\theta$. Now, you see a pulse of radiation; its emitted duration is $\Delta t^{em} = \Delta s/v$; and the duration you observed is shortened by the light-travel time, so that $\Delta t^{obs} = \Delta t^{em} - \Delta s/c$. Putting all of this together, we find

$$\Delta t^{obs} = \frac{2}{\Omega_o\sin\theta}(1-\beta) \simeq \frac{1}{\gamma^2\Omega_o\sin\theta} \qquad (9.7)$$

[1] As usual, $\Omega = eB/\gamma mc = \Omega_o/\gamma$ is the relativistic gyrofrequency.

where in the last step we have used $1/\gamma^2 \simeq 2(1-\beta)$, which is valid when $\gamma \gg 1$.

Thus – we find that the observed pulse has a duration $\simeq 2\pi/\gamma^2\Omega_o$ (converting to Hz); we expect this to be the highest frequency at which there is significant radiated power.

• **Now, do the math.** When the calculation is done more formally, the characteristic frequency for synchrotron radiation turns out to be

$$\nu_c = \frac{3}{4\pi}\gamma^2 \frac{eB}{mc}\sin\theta \tag{9.8}$$

When the power spectrum is evaluated, we find that the radiation can be separated into components which are linearly polarized along and across the magnetic field, *as seen projected on the sky*. These are called $P_{\|}$ and $P_{\perp}$. Considering the strong forward beaming effect, we would expect the component polarized at right angles to the field to dominate (why? Is this clear to you?) The form of the spectrum comes out to be

$$\begin{aligned} P_{\perp}(\nu, E) &= \frac{\sqrt{3}}{2}\frac{e^3 B\sin\theta}{mc^2}\left[F\left(\frac{\nu}{\nu_c}\right) + G\left(\frac{\nu}{\nu_c}\right)\right] \\ P_{\|}(\nu, E) &= \frac{\sqrt{3}}{2}\frac{e^3 B\sin\theta}{mc^2}\left[F\left(\frac{\nu}{\nu_c}\right) - G\left(\frac{\nu}{\nu_c}\right)\right] \end{aligned} \tag{9.9}$$

where

$$G(x) = xK_{2/3}(x)\ ; \ \ F(x) = x\int_x^\infty K_{5/3}(x')dx'$$

and the K's are Bessel functions. Since the $F(x)$ and $G(x)$ functions behave similarly, this verifies that the emissivity polarized transverse to the (projected) magnetic field is much stronger than the emissivity polarized parallel to the field.

Clearly, the total power

$$\begin{aligned} P(\nu, E) &= P_{\perp}(\nu, E) + P_{\|}(\nu, E) \\ &= \sqrt{3}\,\frac{e^3 B\sin\theta}{mc^2}F\left(\frac{\nu}{\nu_c}\right) \end{aligned} \tag{9.10}$$

Both $F(x)$ and $G(x)$ peak at $x \simeq 1$. The asymptotic behavior of $F(x)$ is

$$\begin{aligned} F(x) &\to \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} \qquad x \ll 1; \\ F(x) &\to \left(\frac{\pi x}{2}\right)^{1/2} e^{-x} \qquad x \gg 1 \end{aligned} \tag{9.11}$$

The function $G(x)$ behaves similarly. Remembering that $x = \nu/\nu_c$, this also verifies our guess: the radiation peaks at ν_c and falls off exponentially above this. Further, we note that the particle energy $E = \gamma mc^2$ enters the emissivity expressions (9.9) only through the scaling of the argument, ν/ν_c.

9.3 Spectrum from a distribution of particle energies

The next step is to find the spectrum radiated by a distribution of electron energies, $f(E)$. Consider the total emission, the sum of both polarized components, with total power $P(\nu, E) = P_{\|}(\nu, E) + P_{\perp}(\nu, E)$. We have for the total emissivity per volume,

$$j_{sy}(\nu) = \frac{1}{4\pi}\int P(\nu, E)f(E)dEd\Omega \tag{9.12}$$

We will assume $f(E)$ is isotropic, and integrate over solid angle, in what follows.

Motivated by the observed cosmic ray spectrum, and the photon spectrum seen in many polarized radio sources, we choose the usual power-law particle spectrum,

$$f(E) = f_o E^{-s} \tag{9.13}$$

for the energy range $E_{min} < E < E_{max}$, where $E_{max} \gg E_{min}$ is usually assumed. Note that there must be an E_{min}; (9.13) diverges at low energies. There may well be an E_{max} also – we'll discuss that later.

We put this DF into equation (9.12), and write the single-particle power as $P(\nu, E) = P_o BF(\nu/c_o BE^2)$. From the form of $P(\nu, E)$ and the energy range chosen, note that we expect strong emission in the frequency range

$$\nu_{min} < \nu < \nu_{max} \tag{9.14}$$

where $\nu_{min} = \nu_c(E_{min})$ and $\nu_{max} = \nu_c(E_{max})$. Putting this $f(E)$ into (9.13), we get

$$j_{sy}(\nu) = P_o B f_o \int_{E_{min}}^{E_{max}} E^{-s}F(\nu/c_o BE^2)dE \tag{9.15}$$

Now, by changing the variable of integration from E to $x = \nu/c_o BE^2$, we end up with

$$\begin{aligned} j_{sy}(\nu) =& P_o B^{(s+1)/2} f_o \nu^{-(s-1)/2} \\ &\times \int_{x_{min}}^{x_{max}} x^{(s-3)/2}F(x)dx \end{aligned} \tag{9.16}$$

Now, the integral in (9.16) can be evaluated numerically if $x_{min} \to 0$ and $x_{max} \to \infty$ (which is a reasonable limit for the frequency range in (9.14); Pacholczyk, for instance, has numerical values. Thus, the emissivity in (9.16) can be expressed numerically as

$$\begin{aligned}\epsilon_{sy}(\nu) &= 4\pi j_{sy}(\nu) \\ &= 1.18 \times 10^{-22} a(s) f_o B^{(s+1)/2} \left(\frac{\nu}{2c_1}\right)^{-\alpha}\end{aligned} \tag{9.17}$$

where $c_1 = 6.3 \times 10^{18}$ Hz, $\alpha = (s-1)/2$ is called the *spectral index*, $a(s)$ is an order-unity function (noting the dependence of the x-integral in (9.16) on s), and everything in (9.17) is in cgs units.

Equation (9.17) is good only for the frequency range (9.14). Outside this range, the emissivity will be dominated by that of particles at E_{min} (for $\nu < \nu_{min}$), or of particles at E_{max} (for $\nu > \nu_{max}$). Thus, for $\nu < \nu_{min}$, we expect

$$\text{low } \nu'\text{s}: \quad j_{sy}(\nu) \propto \nu^{1/3} \; ; \tag{9.18}$$

and for $\nu > \nu_{max}$, we expect

$$\text{high } \nu'\text{s}: \quad j_{sy}(\nu) \propto \nu^{1/2} e^{-\nu/\nu_{max}} \; . \tag{9.19}$$

These limits may be repeated in the total spectrum from a plasma – we'll discuss this below.

9.4 Polarization

We noted that, since $P_\perp > P_\parallel$, the radiation from a single particle is linearly polarized. The fractional linear polarization is ususally defined as

$$\pi(\nu) = \frac{P_\perp - P_\parallel}{P_\perp + P_\parallel} \tag{9.20}$$

For a single particle energy, $\pi(\nu) = G(\nu/\nu_c)/F(\nu/\nu_c)$. For a power-law distribution of particle energies, both $P_\perp$ and $P_\parallel$ in (9.20) must be integrated over particle energy; the result is

$$\pi = \frac{\int G(x) E^{-s} dE}{\int F(x) E^{-s} dE} = \frac{s+1}{s+7/3} \tag{9.21}$$

where we have taken $x = \nu/\nu_c$ as above. In evaluating the integrals, we have used the fact that the integrals $\int_0^\infty x^p F(x) dx$ and $\int_0^\infty x^p G(x) dx$ can be expressed in closed form in terms of gamma functions. Thus, a source with $s \simeq 2-3$ will have $\sim 70\%$ polarization.

9.5 Synchrotron self-absorption

This is of course the inverse process – in which a free electron in a magnetic field can absorb a photon. We will treat this by relating the absorption probability to the emissivity, taking stimulated emission into account, using a powerful statistical method developed by Einstein, called Einstein coefficients.[2] In order to do this, we will represent the free electron energy state as a discrete state in a continuum (and after all, even the free electron phase space is quantized, remember), so that the absorption is a transition between a lower state, with energy $E - h\nu$, and momentum $\mathbf{p}(E - h\nu)$, and an upper state with energy E and momentum $\mathbf{p}(E)$.

Referring to the Appendix, we see that the absorption coefficient, at frequency ν, can be written in terms of the populations of all pairs of upper and lower electronic states which are separated by $h\nu$ of energy:

$$\kappa_\nu = \frac{h\nu}{4\pi} \sum_E \left[N(E - h\nu) B_{12} - N(E) B_{21}\right] \tag{9.22}$$

where $N(E) \simeq f(E)dE$ is something like, "the number of electrons in the Eth state", if $f(E)$ is the electron distribution function. Now, we note several facts:

- $B_{12} = B_{21}$ for free electrons, which have the same degeneracy factors $g_1 = g_2$ for the upper and lower states;
- $A_{21} = (2h\nu^3/c^2) B_{21}$;
- $P_{sy}(\nu, E) = h\nu A_{21}$ relates the single particle emissivity to the A_{21} coefficient;

Now, we switch back from describing discrete electron states to a continuous picture:

$$\sum_E n(E) \to \int f(E) dE \to \int f(\mathbf{p}) d^3\mathbf{p}$$

We can thus rewrite (9.22) as

$$\begin{aligned}\kappa_\nu = &\frac{c^2}{8\pi h\nu^3} \int \{f[\mathbf{p}(E - h\nu) - f[\mathbf{p}(E)]\} \\ &\times P[\nu, E(\mathbf{p})] d^3\mathbf{p}\end{aligned} \tag{9.23}$$

Now, we can use the fact that the photon energy should be small compared to the electron energy, $h\nu \ll E$,

[2] We haven't seen these formally; in case you haven't seen them in another class, I'm putting the important discussion in the Appendix to this chapter.

and expand the difference inside the braces in the integrand:

$$f[\mathbf{p}(E - h\nu)] - f[\mathbf{p}(E)] \simeq \frac{h\nu}{c}\frac{df(p)}{dp} \tag{9.24}$$

Finally, since we are still assuming an isotropic particle distribution, we can go back to energy space by noting that $d^3\mathbf{p} = 4\pi p^2 dp = 4\pi E^2 c^{-3} dE$ (where the latter assumes the particles are highly relativistic). This gives us a general expression for the synchrotron absorption from a distribution of electrons,

$$\kappa_\nu = -\frac{c^2}{8\pi\nu^2}\int P(\nu, E) E^2 \frac{d}{dE}\left[\frac{f(E)}{E^2}\right] dE \tag{9.25}$$

Equation (9.25) is still general, except for the assumption of an isotropic particle distribution. Now, if we specify $f(E)$ to be the usual power law, and use the same variable transform as in (9.16), we find the synchrotron self-absorption from a power-law electron distribution:

$$\begin{aligned}\kappa_{sy}(\nu) =& (s+2)\frac{c^2}{8\pi} P_o f_o c_o^{(s+8)/2} \\ & \times B^{(s+2)/2}\nu^{-(s+4)/2}\int x^{-(s+1)/2} F(x) dx\end{aligned} \tag{9.26}$$

The x-integral, again, can be evaluated as a function of s; note, s also appears in the exponents of other parameters in (9.27). If we pick $s = 2.5$ as typical, we can evaluate the constants in $\kappa_{sy}(\nu)$ (using cgs units, still!):

$$\kappa_{sy}(\nu) \simeq 8 \times 10^{-40} f_o B^{(s+2)/2}\left(\frac{\nu}{c_1}\right)^{-(s+4)/2} \tag{9.27}$$

9.6 Total synchrotron spectrum

Finally, we want to consider overall shape of the photon spectrum seen from a synchrotron source. When we asked this question for a thermal source (such as bremmstrahlung), we only had to deal with the radiative transfer. Here, we must also deal with the underlying electron spectrum – as we cannot assume it's thermal.

First, consider the effect of self-absorption on the emergent spectrum from a synchrotron source. We recall the solution to the transfer equation:

$$I_\nu = S_\nu\left(1 - e^{-\tau_\nu}\right) \tag{9.28}$$

where $S_\nu = j_\nu/\kappa_\nu$ and $\tau_\nu = \int \kappa_\nu dx$, integrated through the source. This has the limiting solutions, $I_\nu \to j_\nu x$ for $\tau_\nu \ll 1$ (corresponding to high frequencies for the synchrotron case), and $I_\nu \to S_\nu$ (corresponding to low frequencies). Further, we note that $S_\nu \propto \nu^{5/2}$ from (9.17) and (9.27); since we have assumed a non-Maxwellian electron distribution, we should expect our source function not to be the low-frequency limit of the black body spectrum.

The total spectrum from a source which has $\tau_\nu = 1$ at some observed frequency will thus have a low-frequency range,

$$I_\nu \propto \nu^{5/2} \tag{9.29}$$

and a high frequency range,

$$I_\nu \propto \nu^{-(s-1)/2} \tag{9.30}$$

(this is the optically thin range).

This is not the full answer, however. We pointed out earlier that the electron distribution must have a low-energy cutoff, E_{min} (with critical frequency $\nu_{min} = \nu_c(E_{min})$; and that it may also have a high-energy cutoff, E_{max} (with critical frequency $\nu_{max} = \nu_c(E_{max})$. If we consider these limits, but ignore transfer, then we expect three frequency ranges. Thus, a purely optically thin spectrum will have a low-frequency range,

$$I_\nu \propto \nu^{1/3} \tag{9.31}$$

(why? compare the single-particle spectrum, 9.10 and 9.11). The source will have a mid-frequency range,

$$I_\nu \propto \nu^{-(s-1)/2} \tag{9.32}$$

This source will also have a turnover at high frequencies, due to a cutoff in the electron energy distribution (say at γ_{max}). The most likely spectral form is

$$I_\nu \propto e^{-\nu/\nu_{max}} \tag{9.33}$$

A variant of this can be obtained if the high-energy electron distribution does not cut off abruptly, but more slowly, as is the case in some models of electron aging.

Finally: what might be the cause of low-energy and high-energy cutoffs? The high-energy cutoff is commonly assumed to be due to what's called "spectral aging". That is: from (9.4) or (9.5), recall that the single particle power goes as $P \propto E^2$. Thus, if we start with a power law electron distribution (as in 9.13), the highest energy particles will lose energy the fastest. This leads to a truncation of the electron spectrum, at energies whose synchrotron lifetime equals the age of the source. The low-energy cutoff is harder to specify. It is likely to be due to the fundamental particle acceleration mechanism. We discussed this in Chapter 8; it may be that the low-γ limit of the resonance between Alfven waves and accelerated particles produces this E_{min}.

References

This comes mostly from my own notes. More details can be found in

• Pacholczyk, *Radio Astrophysics*

• Rybicki & Lightmann, *Radiative Processes in Astrophysics*

Key points

• Single particle synchrotron power *and* spectrum;

• Synchrotron emission from a power-law particle DF;

• Synchrotron self-absorption;

• Total synchrotron spectrum: high and low ν cutoffs.

Appendix: Einstein coefficients

I'm taking this directly from Rybicki & Lightman. Remember Kirchoff's law, $j_\nu = \kappa_\nu B_\nu$ (which we saw back in radiative transfer). This relates emission to absorption for a thermal emitter; clearly there must be some relation between emission and absorption at the microscopic level (described by quantum mechanics). To find it, consider transitions between two discrete energy levels of an atom, the first with energy E and statistical weight g_1, the second with energy $E + h\nu_o$ and statistical weight g_2. There are three possible radiative transitions between them:

• **Spontaneous emission** occurs when the upper level spontaneously emits a photon; this occurs whether or not the atom sits in an external radiation field. We define A_{21} as the probability per unit time (sec^{-1}) of this happening (in principle A_{21} could be calculated from quantum physics if we knew all the details of the atom).

• **Absorption** occurs in the presence of photons of energy $h\nu_o$. We define another coefficient, B_{12}, such that $B_{12}J$ is the probability per time for absorption. An important detail here: the spectral line associated with the transition has some finite width, $\delta\nu$, about ν_o (due to energy level uncertainty, doppler and collisional broadening, etc). If $\phi(\nu)$ is the shape of this spectral line, we define $J = \int J_\nu \phi(\nu) d\nu$ as the weighted-mean intensity over the line. Note that, if J_ν varies slowly over the line, $\phi(\nu)$ acts like a delta function.

• **Stimulated emission** also occurs if there are photons of energy $h\nu_o$ around. This wasn't expected classically. Einstein found that it was needed in order to derive Planck's law; we now know it's why masers mase and lasers lase. We define a third coefficient, B_{21}, so that JB_{21} is the probability per time of a stimulated emission event.

The game, then, is to use thermodynamic equilibrium results to find relations between A_{21}, B_{21} and B_{12}. We know three such results:

(i). In TE, the number of radiative transitions into state 1 must equal the number of transitions out of state 1. If n_1 and n_2 are the number of atoms in the two states, we know $n_1 B_{12} J = n_2 A_{21} + n_2 B_{21} J$.

(ii). In TE the ratio of level populations is given by $n_2/n_1 = (g_2/g_1) e^{-h\nu_o/KT}$.

(iii). In TE the photon field is given by the Planck function: $J = B_\nu$ (for $\nu = \nu_o$).

These three facts are enough: we can solve the equations to show that the Einstein coefficients must satisfy

$$g_1 B_{12} = g_2 B_{21} \; ; \quad A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{9.34}$$

That's the main result. We can – in principle – use quantum physics to determine A_{21} for any given transition; then (9.34) tells us what the B's must be.

The Einstein coefficients can also be used to determine the emission and absorption coefficients. Going back to the definitions (chapter 3), we can build

$$j_\nu = \frac{h\nu_o}{4\pi} n_2 A_{21} \phi(\nu) \tag{9.35}$$

and

$$\kappa_\nu = \frac{h\nu_o}{4\pi} \left(n_1 B_{12} - n_2 B_{21} \right) \phi(\nu). \tag{9.36}$$